

Comparative Statics of General Equilibrium Asset Prices

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Abstract

I study comparative statics of asset prices in a representative-agent model where dividends are vector-geometric Brownian motions. Due to wealth effects, the equilibrium relative price of a security may vary with the current realization of a component of the Brownian vector even when its dividend is independent of that component. I examine analytically an element of wealth effects that has hitherto been ignored by the literature. Changes in wealth do not operate only through changes in risk aversion. They alter also the riskiness of a security by changing the correlation between its payoff and the marginal utility of equilibrium consumption. This enhances the extent to which market-clearing leads to endogenously-generated correlation across asset prices and returns, over and above that induced by correlation between payoffs, giving the appearance of “contagion”. I also establish a necessary and sufficient condition for the securities market to be dynamically complete when it is potentially so. It applies for any utility function of the representative agent.

Keywords: General Equilibrium, Asset Prices, Comparative Statics, Contagion, Dynamically Complete Markets

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1 Introduction

Within the continuous-time financial asset-pricing literature, the work-horse model has been that of a one-good, pure exchange, finite-horizon economy with identical, price-taking consumers in which economic activity takes place in the time interval $[0, T]$. The single good is (costlessly) produced by N distinct productive units, the productivity of each fluctuating stochastically through time. The amount of the good that has been accumulated at any point in time is observed by everyone. Yet, the good can be consumed only at the terminal date T , the usual interpretation being the famous Lucas' tree (as in Lucas [28]). A crop is growing stochastically on trees, being observed at any time but ripe for consumption only at T , the point at which it will be consumed and the world will end. The production process of this crop is entirely exogenous: no resources are utilized and there is no possibility of affecting the output of any unit at any time. In fact, in this economy, the magnitude of the crop at time t plays the role of an "information process;" it is monitored by all individuals who are continuously revising their beliefs about its terminal payoff.

The trading structure consists of $N + 1$ perfectly divisible securities, which are continuously and frictionlessly traded in a market. The typical security $n \in \{1, \dots, N\}$ represents one equity share (termed "stock") in the n th productive unit. It entitles its holder at time T to receive the total amount that will have been produced by that unit. The remaining security is a promissory note (termed a "bond") that pays one unit of the consumption good at time T with certainty. This zero-coupon bond is in zero net supply while each stock's net supply is one unit. The structure of individual preferences is such that the representative agent is assumed to have a state-independent, twice-continuously differentiable everywhere on its domain, von-Neumann Morgenstern utility function of date- T consumption, $u : \mathbb{R}_{++} \mapsto \mathbb{R}$. It will be taken to be everywhere in its domain strictly increasing and concave.

As is well-known, in this economy, the equilibrium price of the n th risky security at any time $t \in [0, T)$ is the current expectation of its terminal dividend, valued at the equilibrium marginal utility of the representative agent at the terminal date. Of course, marginal utilities are not observable in practice and securities are priced with respect to a numeraire, such as dollars. Fortunately, when the underlying information process evolves on a \mathcal{F}_t , the choice of numeraire is essentially arbitrary because the equilibrium market-clearing condition depends only on the relative prices of the securities and consumption and does so node (ω, t) by node on \mathcal{F}_t , not across nodes.

We can take, therefore, consumption as the numeraire so that its price will be identically 1 at any time $t \in [0, T)$. Then, letting $W(\cdot)$ denote the representative agent's wealth process

(in units of consumption) and $\mathcal{I}(\omega, t)$ the current realization of the information process, the equilibrium processes of the n th risky security and of the bond are given, respectively, by

$$\begin{aligned} P_n(\mathcal{I}(\omega, t)) &= \mathbb{E}[u'(W(\mathcal{I}(\omega, T))) D_n(\mathcal{I}(\omega, T)) | \mathcal{I}(\omega, t)] \\ P_0(\mathcal{I}(\omega, t)) &= \mathbb{E}[u'(W(\mathcal{I}(\omega, T))) | \mathcal{I}(\omega, t)] \end{aligned}$$

at any $t \in [0, T]$. Derivations of this are provided, for example, by Cox *et al.* [16], Bick [9], Cochrane *et al.* [15], when the securities market is dynamically complete, and by Raimondo [33] and Anderson and Raimondo [5], when it is allowed to be dynamically incomplete.¹ Allowing the individuals to have non-identical preferences for consumption, the pricing formula takes still the same basic form. The individual marginal utilities are now taken at the equilibrium consumptions of the agents, which are determined endogenously as part of the equilibrium (see, for instance, Duffie and Zame [19] or Anderson and Raimondo [4]).

Even, though, this model has been extensively used in the literature, the dynamics of these prices with respect to the currently observed information have not been studied, at least not analytically and not to any considerable generality. And this is the task of the present paper. To be more specific, I examine the dynamics of the equilibrium price of the typical risky security relative to the price of the bond,

$$p_n(\mathcal{I}(\omega, t)) = \frac{P_n(\mathcal{I}(\omega, t))}{P_0(\mathcal{I}(\omega, t))}$$

with respect to the current realization in the typical dimension of the information process, $\mathcal{I}_k(\omega, t)$. Whether these dynamics are monotone is the most fundamental comparative statics question. For if so, the equilibrium relative prices of the securities vary predictably in response to changes in current information about future dividends. This would, for instance, greatly facilitate econometric analysis since the realized path of the underlying stochastic process (representing the primitive sources of uncertainty) could be recovered from the path

¹See equation (4) in Bick [9] who chose the bond as the numeraire. That is, $P_0 = 1$ at any $t \in [0, T]$ while the price of consumption is given by $P_c(\mathcal{I}(\omega, t)) = \mathbb{E}[u'(W(\mathcal{I}(\omega, T))) | \mathcal{I}(\omega, t), \mathcal{F}_t]$. In Cox *et al.* [15], I am referring to the last term on the right-hand side of equation (38). It is this term that prices real assets, claims to some amount of the underlying consumption good. The first two terms allow for the pricing of general financial assets, including options and futures. More precisely, claims that pay $\Theta(W(T), Y(T))$ if some underlying variables do not leave a certain region before the maturity date T and $\Psi(W(t), Y(t), t)$ every time t they do, otherwise. Notice also that $J(W(t), Y(t), t)$ is the equilibrium indirect utility at t . It depends on the state variable $Y(t)$ as the authors allow for the direct utility to be state-dependent, a level of generality beyond the scope of my study. Observe also that, in my exposition, $J_W(W(t), Y(t), t)$, the marginal utility at the current date t , is identically 1 for all $t \in [0, T]$ for it prices consumption, the numeraire.

of equilibrium relative asset prices.

Even though $p_n(\mathcal{I}(\omega, t))$ has actually been derived in closed form in the setting I study (in particular, by Raimondo [33]), determining its basic comparative statics' properties is not straightforward for two reasons. First, by the quotient rule, the derivative $\frac{\partial p_n(\mathcal{I}(\omega, t))}{\partial \mathcal{I}_k(\omega, t)}$ will be given as the sum of two terms which may well be of opposite sign. Second, and more important, even the absolute price of the security itself may exhibit complex dynamics. An increase in the n th terminal dividend increases the terminal-period consumption, reducing its marginal utility. Since $P_n(\mathcal{I}(\omega, t))$ is given by the expectation of the product of $D_n(\mathcal{I}(\omega, T))$ with this marginal utility, it need not increase when the dividend increases.²

To study these dynamics, I will take the production process of the n th unit, $\{D_{n,t}\}$, to be following a K -dimensional geometric Brownian motion with a drift parameter μ_n and instantaneous variance-covariance of return σ_n . Formally,

$$dD_{n,t} = \mu_n D_{n,t} dt + \sigma_n D_{n,t} d\beta_t$$

where $\mu_n \in \mathbb{R}$ and $\sigma_n^\top \in \mathbb{R}^K$ are constant, while $\{\beta_t\}$ is a K -dimensional standard Brownian motion with respect to a filtration $\{\mathcal{F}_t : t \in [0, T]\}$ on a probability space (Ω, \mathcal{F}) . That is, letting $\mathcal{I}(\omega, t) = (t, \beta(\omega, t))$ for $t \in [0, T]$, the dividend of the typical risky security will be given by

$$D_n(\mathcal{I}(\omega, t)) = 0 \quad \text{for } t < T \quad D_n(\mathcal{I}(\omega, T)) = e^{\mu_n T + \sigma_n^\top \beta(\omega, T)}$$

The same specification can be found, for instance, in Merton [30], Bick [9], He and Leland [23], Cox *et al.* [16], Raimondo [33], Anderson and Raimondo [5], and Cochrane *et al.* [15].³

My most general result refers to the dynamics of the relative price of the typical risky security with respect to changes in the current realization of the entire Brownian vector, $\beta(\omega, t)$. It states that the inner-product of the vector σ_n , the n th row of the dispersion matrix Σ of the underlying Brownian process, with the gradient vector $\nabla_{\beta(\omega, t)} p_n(\omega, t)$, the (transpose of the) n th row of the dispersion matrix of relative prices, is always positive (Theorem ??). It follows immediately that, if the n th terminal dividend is correlated with

²Raimondo [33] offers an example, with log-utility, in which $p_n(\mathcal{I}(\omega, t))$ is *independent* of $D_n(\mathcal{I}(\omega, T))$. All of the adjustment in the relative price, that is needed to clear the markets, occurs in the price of the bond.

³Bick [9] considers a more general diffusion process $\{D_t\}$, one that is positive and of the same form but not necessarily a geometric Brownian motion. That is, μ and σ need not be constants. They are both scalars as he studies only the case $N = K = 1$. The same is true for He and Leland [23]. In Merton [30], the diffusion process is equally general as in these two papers but there are multiple securities and Brownian motions. Even though he calls his typical diffusion process the price of his typical security, it is really its dividends since consumption is the numeraire. Cox *et al.* [16] allow for even more general dividend processes.

only Brownian motion, the relative price of the n th risky security is always monotone in the realization of this Brownian motion. This applies, of course, on all N securities in two important cases: when there is a single source of uncertainty affecting all terminal dividends ($K = 1$) or when the matrix Σ is diagonal.⁴

I proceed to study the derivative $\frac{\partial p_n(\omega, t)}{\partial \beta_k(\omega, t)}$ when the terminal-period dividend $D_n(\omega, T)$ is independent of the $\beta_k(\omega, T)$ component of the Brownian vector ($\sigma_{nk} = 0$). To the extent that $\beta_k(\omega, T)$ does affect some of the other $N - 1$ terminal-period dividends or the agent's terminal-period endowment, it induces wealth effects which may require adjustments in the equilibrium relative price of the n th security. Evidently, the dynamics of the equilibrium relative price are rich and the cross-derivative will not be zero apart from quite unusual cases. The comparative statics of equilibrium relative prices are complex since changes in the underlying Brownian process induce wealth effects which alter not only the agent's risk aversion but also the "riskiness" of a security. The latter effect operates through changing the covariance between the dividend of the security and the marginal utility of the agent. It turns out to be fundamental for the dynamics of the equilibrium relative price and is examined here in detail.

The complexity of the dynamics of relative prices can be demonstrated analytically in some settings. In fact, it can be shown to be unusual for the relative price of the n th security to not vary with the realization of a Brownian motion even when its terminal dividend is independent of that Brownian component. And this is true even when the agent exhibits constant absolute risk aversion (CARA). The relative price will typically respond to changes in the current realization of a Brownian dimension even though the security's payoff is independent of it. In fact, it will not do so only if this Brownian motion affects the agent's terminal-period wealth through channels that are independent from those through which the wealth is affected by the Brownian dimensions that are correlated with the dividend (Proposition 3.1.1). Propositions 3.1.2 and 3.2.1 establishes that this separation is also necessary. For they describe settings in which it is violated and the n th relative price varies with the realization of the k th Brownian motion; indeed, it does so monotonically.

Regarding more general attitudes towards risk, Propositions 3.1.3 and 3.2.2 construct situations in which the relative price of a security will vary monotonically with a Brownian motion not correlated with its dividend, under any decreasing absolute risk aversion (DARA) utility. It should be noted that the former proposition includes the case in which the dis-

⁴The $N \times K$ matrix Σ has σ_n^\top as its n th row. A diagonal Σ is necessarily square: $N = K$, $\sigma_{kk} \neq 0$, and $\sigma_{nk} = 0$ for $n \neq k$.

ersion matrix Σ is diagonal and the agent’s terminal-period endowment is deterministic. Admittedly, this is the most restrictive against cross-correlations in prices environment. This result really shows, therefore, that endogenously-generated correlation across asset prices and returns, over and above that induced by correlation between payoffs, is a generic equilibrium phenomenon.

The possibility for a “common factor” or “contagion” in relative asset prices (and, thus, returns) to emerge, when there is no common factor in cash flows, is well-known but has not been demonstrated before analytically in a general equilibrium model. It is noted, for example, in Raimondo [33] and Anderson and Raimondo [?] but no formula is given for the cross-derivative. Kodres and Pritsker [24], Kyle and Xiong [25], and Lagunoff and Schreft [26] show that “contagion” can obtain as a wealth effect in rational expectations equilibria. These are not general equilibrium models, however, as some market participants are not rational (the former two models require the presence of noise traders; the latter of irrational ones). Similarly, Aliprantis *et al.* [1] establish contagion equilibria in a monetary model where players act strategically.

The paper that is closest to the present is Cochrane *et al.* [15] who study asset-price and return dynamics in a representative-agent model with two Lucas [28] trees. Each tree’s dividend stream follows a geometric Brownian motion while the agent has log-utility and consumes the sum of the two dividends. This study encompasses a large collection of variables of interest with closed-form solutions given for absolute prices, expected returns, volatilities, correlations, etc. However, the solution method cannot be applied beyond log-utility and two trees (it depends fundamentally upon the dividend-consumption share being the unique state variable) while the dynamics are examined numerically. More importantly, they are given with respect to the dividend-share rather than the underlying stochastic process representing the uncertainty.

Through numerical estimation, the authors show that the prices and returns of the two assets can be positively contemporaneously correlated even though the underlying dividends are independent. Proposition 3.1.3 confirms the positive correlation of relative prices, analytically and for any DARA utility. The comparative statics with respect to the dividend-share seem to express the underlying market-clearing intuition. If there is a dividend shock, the representative agent wants to rebalance, to spread some of her now larger wealth across both trees. Since she has to hold the fixed assets supply in equilibrium, however, she cannot rebalance, so asset prices and expected returns must adjust. Given a positive dividend shock on tree one, the dividend-share of asset one increases while that of asset two falls. For the agent

to become willing to hold asset two in smaller proportion, it must be made less attractive. That is, its price must rise so that its expected return falls. Equivalently, trying but not being able to rebalance some of her larger wealth away from asset one, the agent pushes up the price of asset two.

Although correct, this intuition fails to distinguish between two separate channels through which shocks to current wealth affect market clearing, via changing the agent's risk aversion but also through altering her perception of the "riskiness" of a security. The dynamics of the former mechanism are well-known and straightforward. Those of the latter have not, to the best of my knowledge, hitherto been analyzed by the finance literature and are complex. Under DARA and independent dividend streams, the two mechanisms operate in the same direction which, however, is not universally the case. The operation of the asset-riskiness effect on relative prices can be isolated under CARA since the risk-aversion channel of wealth effects leaves relative prices unchanged. As Proposition 3.1.2 shows, it can easily lead to negative correlation.

The comparative statics formulae and results in this paper lend themselves easily to empirical testing; they can be calculated numerically for any sets of the parameters of the model. The fact that the equilibrium relative prices of assets and asset returns should be correlated, even when their underlying dividends are independent, has important implications for asset-pricing. In particular, it raises questions about the large body of work that focusses on partial equilibrium analysis, treating a small number of securities in isolation from the rest of the market or modeling the equilibrium price process of an asset as a relation that depends only on those sources of uncertainty that directly affect its payoff. Even though the empirical finance literature has focused attention on contagion across national or regional stock markets, a few papers have established that the magnitude of the correlations across asset returns cannot be explained by covariances between the sources of uncertainty that determine their respective payoffs alone. For example, Gropp and Moerman [22] identify within country contagion among large European bank stocks. Driessen *et al.* [17], Lopez and Walter [27], and Moskowitz [32] find evidence that the risk premia are better represented by covariances with the implied market- than by own-variances.

In the model examined here, correlations across asset returns arise endogenously and are stochastic, even though the covariance coefficients of the dividends are constant. There is ample evidence in the empirical finance literature that correlations across asset returns are stochastic. Bollerslev *et al.* [11] present reasonable estimates for a trivariate (U.S. Treasury bills, bonds, and stocks) CAPM model in support of the conclusion that the conditional

covariance matrix of asset returns is strongly autoregressive. Other studies have documented cross-sectional relations between risk (measuring generally a stock's risk as the covariance between its return and one or more variables) and expected returns on common stocks. For example, the expected return on a stock has been found to be related to covariances between its return and (i) the return on the market portfolio (Black *et al.* [10], Fama and Macbeth [20]), (ii) factors extracted from multivariate time series of returns (Roll and Ross [34]), (iii) macroeconomic variables (Chen *et al.* [14]), and (iv) aggregate consumption (Breedon *et al.* [13]). See also Andersen *et al.* [3], Alizadeh *et al.* [2], Bansal and Yaron [6], Bansal *et al.* [7], Tauchen [?], Brandt and Diebold [12], and Schwert and Seguin [35] for more recent work.

My analysis contributes also to the literature on equilibrium in continuous-time finance models with a single agent. Existence has been established in a number of papers (Bick [8], He and Leland [23], Cox *et al.* [16], Duffie and Skiadas [18], Raimondo [33]) but very little is known regarding the question of whether the equilibrium is dynamically complete, if it exists. Apart from Raimondo [33], all papers deal only with the case in which markets are potentially dynamically complete ($N = K$).⁵ When there are more than one sources of uncertainty ($K > 1$), they show existence of equilibrium by computing the candidate equilibrium price process explicitly and checking that it is dynamically-complete. However, such computation can be done usually only for a very small set of parameter values (in fact, none of the results rules out the possibility that this set has Lebesgue measure zero).⁶ For $N = K$, I show that the equilibrium pricing process is in fact dynamically complete, for any parameter values and whatever the utility function of the representative agent, if and only if the dispersion matrix Σ is diagonal (Theorem 4.1). Even though rather widely asserted, this has not been shown before in the literature.

To complete introducing my results, it should be noted that there is nothing pathological about the model I study. The utility functions can include any twice continuously-differentiable, state-independent functions representing non-satiated preferences and risk-aversion. The dividend processes are geometric Brownian motions. Both are central benchmarks in continuous-time finance.⁷ The agent is endowed with a flow rate of consumption

⁵Raimondo [33] derives the equilibrium price process directly from the primitives of the economy and in closed form. His existence theorem and equilibrium pricing formulae apply equally well when markets are potentially dynamically-complete ($N = K$) and when they are necessarily dynamically-incomplete ($N < K$).

⁶The issue of dynamic completeness of the equilibrium price process is even more serious in the continuous-time models with many agents. Apart from Anderson and Raimondo [4], the literature assumes, in various forms, that the candidate equilibrium price process is dynamically-complete and proceeds to show that is in fact an equilibrium. See Anderson and Raimondo [4] for a detailed discussion.

⁷Some quite general state-dependence of the utility functions can also be allowed, as long as the dependence enters through the process $\{(t, \beta(\omega, t))\}$; that is, as long as it is measurable in the components of the

on $[0, T)$ and a lump sum at T (described in many models as a bequest). All N risky securities pay a lump dividend, in units of consumption, at time T (which can be viewed also as the present value of their stream of future dividends in an infinite-horizon framework). These elements define a family of continuous-time securities markets that has been one of the standards in the literature (see, for example, Merton [29]). The particular economy examined here allows for completeness of exposition without imposing significant limitations on the analysis. Within the family of markets described above, the equilibrium relative price process will have to be of the same qualitative form as the one whose comparative statics I examine.

The remaining of the paper is organized as follows. The next section investigates the comparative statics of the typical relative price with respect to the typical Brownian motion, its emphasis being on economic intuition and interpretation. Sections 3 and 4 presents, respectively, the results regarding the dynamics of the relative prices and dynamic completeness. All proofs have been delegated to the Appendix.

2 The Mechanics of Comparative Statics

Consider a representative agent economy in which trade and consumption occur over the compact time interval $[0, T]$.⁸ This interval is endowed with a measure λ that agrees with the Lebesgue measure on $[0, T)$ while $\lambda(\{T\}) = 1$. The information structure is represented by a filtration $\{\mathcal{F}_t : t \in [0, T]\}$ on a probability space $(\Omega, \mathcal{F}, \mu)$ and a standard K -dimensional Brownian motion $\beta = (\beta_1, \dots, \beta_K)^T \in \mathbb{R}^K$. The agent has an additively-separable, time-independent utility function. Given a measurable consumption function $c : \Omega \times [0, T] \rightarrow \mathbb{R}_{++}$, her utility function is given by

$$U(c) = \mathbb{E}_\mu \left[\int_0^T u_1(c_t) dt + u_2(c_T) \right]$$

where the twice continuously-differentiable functions $u_i : \mathbb{R}_{++} \rightarrow \mathbb{R}$, $i \in \{1, 2\}$, satisfy $u'_i(\cdot) > 0$ and $u''_i(\cdot) < 0$. Her endowment process is constant except for the terminal period T . It is given by $e : \Omega \times [0, T] \rightarrow \mathbb{R}_+$, with $e(\omega, t) = 1$ for any $(\omega, t) \in \Omega \times [0, T)$ and $e(\omega, T) = \rho(\beta(\omega, T))$ for some continuous function $\rho : \mathbb{R}^K \rightarrow \mathbb{R}_+$.

There is a “bond” (in zero net supply) which pays in units of consumption. Its payoff

Brownian vector.

⁸The ensuing description borrows heavily from Raimondo [33] as well as Anderson and Raimondo [5].

is given by $B : \Omega \times [0, T] \rightarrow \mathbb{R}_+$ with $B(\omega, T) = 1$ and $B(\omega, t) = 0$ at any $t \in [0, T)$. There are also J securities (each in net supply of one unit) with $J \leq K$. Each stock pays off only at time T and the terminal-period dividends follow simple geometric Brownian motions. Taking Σ to be a $J \times K$ matrix with σ_j^\top its j th row, let the dividend of the j th stock be $D_j(\omega, T) = e^{\mu_j T + \sigma_j^\top \beta(\omega, T)}$: $\mu_j \in \mathbb{R}_+$, $\sigma_j \in \mathbb{R}^K$. Raimondo [?] shows that there exists an equilibrium price process for this economy.⁹ It can be obtained in terms of the agent's utility function, her terminal-period endowment, and the current realization of the sources of uncertainty as given by the vector $\beta(\omega, t)$. The equilibrium process consists of (i) a stochastic process for the vector of the J stock prices, $P(\omega, t) = [P_j(\omega, t)]_{j=1, \dots, J}$, and the price of the bond, $P_0(\omega, t)$, and (ii) a stochastic consumption price process $P_C(\omega, t)$. All $J + 1$ asset prices are continuous, square-integrable martingales with respect to the filtration $\{\mathcal{F}_t\}$. They are defined by¹⁰

$$\begin{aligned}
 P_j(\omega, t) &= \mathbb{E}_{\mu|\beta(\omega, t)} [u'_2(F(\omega, T)) D_j(\omega, T)] = \int_{\mathbb{R}^K} u'_2(F(\omega, t, \mathbf{x})) e^{\mu_j T + \sigma_j^\top (\beta(\omega, t) + \sqrt{T-t}\mathbf{x})} d\Phi(\mathbf{x}) \\
 P_0(\omega, t) &= \mathbb{E}_{\mu|\beta(\omega, t)} [u'_2(F(\omega, T))] = \int_{\mathbb{R}^K} u'_2(F(\omega, t, \mathbf{x})) d\Phi(\mathbf{x}) \\
 P_C(\omega, t) &= \begin{cases} u'_1(1) & t \in [0, T) \\ u'_1(F(\omega, T, \mathbf{0})) & t = T \end{cases}
 \end{aligned}$$

where

$$\begin{aligned}
 F(\omega, T) &= \rho(\beta(\omega, T)) + \sum_{i=1}^J D_i(\beta(\omega, T)) \\
 F(\omega, t, \mathbf{x}) &= \rho\left(\beta(\omega, t) + \sqrt{T-t}\mathbf{x}\right) + \sum_{i=1}^J D_i(\omega, t, \mathbf{x})
 \end{aligned}$$

⁹For his existence theorem, Raimondo [?] imposes also three assumptions that are not included in the setting described above. Specifically, the utility functions $u_i(\cdot) : i \in \{1, 2\}$ are assumed to be bounded below: $\exists K > -\infty$ s.t. $u_i(c) > K \forall c \in \mathbb{R}_{++}$. Moreover, in order to not have to handle genericity considerations on existence, a short-sale constraint is introduced: $\exists M > 0$ s.t. the agent is not permitted to hold less than $-M$ units of any of the $J + 1$ traded assets. Finally, the terminal-period endowment function is taken to satisfy $0 \leq \rho(\mathbf{x}) \leq r + e^{r|\mathbf{x}|}$ for some $r \in \mathbb{R}_+$ and $\forall \mathbf{x} \in \mathbb{R}^K$. Anderson and Raimondo [?] show that the first two assumptions are not necessary for existence of equilibrium in a model which nests the one examined here. As for the third condition, it is satisfied by any bounded-above function $\rho(\cdot)$. My results *per se* do not depend upon any assumptions other than the ones already stated in the text. Additional conditions, that may be necessary for an existence proof, are not really relevant for my comparative statics analysis. If an equilibrium price process does indeed exist, the equilibrium relative prices have to be as in (??) and this is where I begin.

¹⁰See Theorem 1 in Raimondo [?] or Theorem 2.1 in Anderson and Raimondo [?].

is the terminal-period wealth (in units of consumption) given, respectively, unconditionally and conditionally on the realization $\beta(\omega, t)$ of the Brownian process while $\Phi(\cdot)$ is the cumulative distribution function for the standard K -dimensional normal. For $\mathbf{x} \sim \mathbf{N}(\mathbf{0}, \mathbf{I}_K)$, $D_i(\omega, t, \mathbf{x}) = e^{\mu_j T + \sigma_j^\top (\beta(\omega, t) + \sqrt{T-t} \mathbf{x})}$ depicts a $\beta(\omega, t)$ -conditional future realization of the i th terminal-period dividend.

In this paper, my focus is on the comparative statics, in equilibrium, of the relative price

$$p_n(\omega, t) = \frac{P_n(\omega, t)}{P_0(\omega, t)} = \frac{\mathbb{E}_{\mathbf{x}} [u'(W(\omega, t, \mathbf{x})) D_n(\omega, t, \mathbf{x})]}{\mathbb{E}_{\mathbf{x}} [u'(W(\omega, t, \mathbf{x}))]}$$

of the typical risky security with respect to changes in $\beta_k(\omega, t)$, the current at $t \in [0, T)$ realization of the typical Brownian motion. More specifically, I am interested in identifying conditions on the economic primitives of the model under study that suffice for $p_n(\omega, t)$ to be monotone in $\beta_k(\omega, t)$. To this end, the building block for my analysis will be a result that holds universally across the space of economic primitives. This is the fact that the inner product of the n th row of the dispersion matrix Σ with the (transpose of the) corresponding row of the dispersion matrix of the relative prices $\left\{ \frac{\partial p_n(\omega, t)}{\partial \beta_k(\omega, t)} \right\}_{k=1, \dots, K}$ is non-negative. More precisely, it is strictly positive unless the n th terminal dividend does not vary with any of the sources of uncertainty in the model.

Theorem 2.1 *Suppose that, at the node $(\omega, t) \in \Omega \times [0, T)$, the following are true.*

(i) $u'(W(\omega, t, \mathbf{x})) > 0 \forall \mathbf{x} \in \mathbb{R}^K$

(ii) *Fixing $\beta_{-k}(\omega, t)$ and viewing*

$$u'(W(\beta_k(\omega, t), \mathbf{x}; \beta_{-k}(\omega, t))) e^{\mu_n T + \sigma_n^\top ((\beta_{-k}(\omega, t), \beta_k(\omega, t)) + \sqrt{T-t} \mathbf{x})}$$

as a function $\mathbb{R} \times \mathbb{R}^K \mapsto \mathbb{R}$ of $\beta_k(\omega, t)$ and \mathbf{x} , it satisfies the conditions for Lemma A.1 to apply.

Then,

$$\sum_{k=1}^K \sigma_{nk} \frac{\partial p_n(\omega, t)}{\partial \beta_k(\omega, t)} \geq 0$$

with equality only if $\sigma_n = \mathbf{0}$.

Clearly, the required conditions for this theorem to apply extremely mild; there are met by all utility functions that are generally of interest in continuous-time finance. Yet, the

result refers to the typical row of the dispersion matrix of relative prices, not to its typical element. In fact, a claim of such generality cannot be obtained for the typical equilibrium relative price because, as the subsequent analysis will show, the dynamics of the relative price with respect to the current realization of the typical dimension of the underlying stochastic process are quite complex, surprisingly so in some situations. This section attests to the richness of these dynamics by means of describing the constituent parts of their generating mechanism.

Towards an overview of this mechanism, let us begin by observing that the (absolute) equilibrium price of the n th risky security can be expressed also as follows

$$P_n(\omega, t) = \text{Cov}_{\mathbf{x}}[u'(W(\omega, t, \mathbf{x})), D_n(\omega, t, \mathbf{x})] + p_0(\omega, t) \mathbb{E}_{\mathbf{x}}[D_n(\omega, t, \mathbf{x})]$$

As a consequence, its derivative with respect to the current realization of the k th Brownian motion can be written as

$$\begin{aligned} \frac{\partial P_n(\omega, t)}{\partial \beta_k(\omega, t)} &= \frac{\partial \text{Cov}_{\mathbf{x}}[u'(W(\omega, t, \mathbf{x})), D_n(\omega, t, \mathbf{x})]}{\partial \beta_k(\omega, t)} \\ &\quad + \mathbb{E}_{\mathbf{x}}[D_n(\omega, t, \mathbf{x})] \frac{\partial P_0(\omega, t)}{\partial \beta_k(\omega, t)} + \sigma_{jk} \mathbb{E}_{\mathbf{x}}[D_n(\omega, t, \mathbf{x})] P_0(\omega, t) \end{aligned} \quad (1)$$

On the other hand, that of the equilibrium price of the bond is given by

$$\frac{\partial P_0(\omega, t)}{\partial \beta_k(\omega, t)} = \mathbb{E}_{\mathbf{x}} \left[u''(W(\omega, t, \mathbf{x})) \frac{\partial W(\omega, t, \mathbf{x})}{\partial \beta_k(\omega, t)} \right] \quad (2)$$

whereas, by Lemma A.2 of the Appendix, the equilibrium relative price of the n th stock can be written as

$$p_n(\omega, t) = \mathbb{E}_{\mathbf{x}}[D_n(\omega, t, \mathbf{x})] \frac{\mathbb{E}_{\mathbf{x}}[u'(W(\omega, t, \mathbf{x} + \sqrt{T-t}\sigma_n))]}{\mathbb{E}_{\mathbf{x}}[u'(W(\omega, t, \mathbf{x}))]} \quad (3)$$

In words, these equations depict the following relations. At the arbitrary node (ω, t) , given an arbitrary realization $\beta(\omega, t)$ of the underlying stochastic process, exchanging one unit of the bond for one unit of the stock increases the currently (i.e. \mathcal{F}_t -conditional) expected terminal-period wealth by the currently expected terminal dividend of the security, $\mathbb{E}_{\mathbf{x}}[D_n(\omega, t, \mathbf{x})] = e^{\mu_n T + \sigma_n^T (\beta(\omega, t) + \frac{(T-t)\sigma_n}{2})}$. In terms of terminal-period wealth, therefore, one unit of the n th stock is equivalent to $e^{\mu_n T + \sigma_n^T (\beta(\omega, t) + \frac{(T-t)\sigma_n}{2})}$ units of the bond. In terms of marginal utility (which is what matters in general equilibrium pricing), however, the corresponding equivalence requires also that any realization $\sqrt{T-t}\mathbf{x} \sim \mathcal{N}(0, (T-t)I_K)$ of

the future increment $\beta(\omega, T) - \beta(\omega, t)$ of the underlying stochastic process gets translated by the quantity $(T - t)\sigma_n$.

The Own-Dividend Effect

Other things remaining equal, a change $d\beta_k(\omega, t)$ in the k th component of $\beta(\omega, t)$ alters by $\sigma_{nk}d\beta_k(\omega, t)$ the \mathcal{F}_t -conditional drift, $\mu_n T + \sigma_n^T \beta(\omega, t)$, of the underlying stochastic process that determines the n th terminal dividend.¹¹ The \mathcal{F}_t -conditional expectation of the terminal dividend itself, then, changes by $\sigma_{nk} \mathbb{E}_{\mathbf{x}} [D_n(\omega, t, \mathbf{x})] d\beta_k(\omega, t)$. Suppose now that $\beta_k(\omega, t)$ increases. If $\sigma_{nk} > 0$ ($\sigma_{nk} < 0$), the currently expected terminal dividend will be higher (lower). Due to non-satiation ($u'(\cdot) > 0$), this increases (decreases) the willingness of the agent to hold the n th risky security. As she must, though, continue to hold its net supply in equilibrium, the (absolute) price of the security must rise (fall) exactly by $\sigma_{nk} P_0(\omega, t) \mathbb{E}_{\mathbf{x}} [D_n(\omega, t, \mathbf{x})] d\beta_k(\omega, t)$, which is the change in the \mathcal{F}_t -conditional drift of the underlying stochastic process in units of the bond. Henceforth, I will be referring to this as the *own-dividend effect* of $d\beta_k(\omega, t)$ on the n th equilibrium price. It is depicted by the third term on the right-hand side of (1).

The Wealth Effect

For any future realization $\sqrt{T - t}\mathbf{x}$ of the stochastic process $\beta(\omega, T) - \beta(\omega, t)$, a change in $\beta_k(\omega, t)$ corresponds to revealing information that changes also the \mathcal{F}_t -conditional expected terminal dividend of any security $n' \in \{1, \dots, N\}$ by $\sigma_{n'k} \mathbb{E}_{\mathbf{x}} [D_{n'}(\omega, t, \mathbf{x})] d\beta_k(\omega, t)$. These changes along with that in the terminal-period endowment, $de(\beta(\omega, t) + \sqrt{T - t}\mathbf{x})$, give the corresponding change in the \mathcal{F}_t -conditional terminal-period wealth. Other things remaining unchanged, the agent's risk aversion ($u''(\cdot) < 0$) induces an opposite change in marginal utility, which will be called from now on the *wealth effect* of $d\beta_k(\omega, t)$.

Regarding the equilibrium price of the bond, this effect is given by equation (2). With respect to the equilibrium price of the n th risky security, it is given by the second term on the right-hand side of (1). Clearly, the direction of the wealth effect is the same on either price. In fact, this is true also for its magnitude. The two terms differ by the proportionality constant needed to convert units of the stock into units of the bond, in terms of \mathcal{F}_t -conditional expected terminal-period wealth.

¹¹“Other things remaining equal” (or similar expressions) refer henceforth to the current realizations of the remaining $K - 1$ sources of uncertainty, $\{\beta_m(\omega, t)\}_{m \in \{1, \dots, K\} \setminus \{k\}}$.

To identify the effect on the relative equilibrium price of the n th risky security, consider its derivative

$$\frac{\partial p_n(\omega, t)}{\partial \beta_k(\omega, t)} = \frac{1}{P_0(\omega, t)} \left[\frac{\partial P_n(\omega, t)}{\partial \beta_k(\omega, t)} - p_n(\omega, t) \frac{\partial P_0(\omega, t)}{\partial \beta_k(\omega, t)} \right] \quad (4)$$

Using equations (2) and (3) and the second term on the right-hand side of (1), it is straightforward to verify that the wealth effect on the relative price is given by

$$\begin{aligned} & \left(\frac{\mathbb{E}_{\mathbf{x}} [D_n(\omega, t, \mathbf{x})] - p_n(\omega, t)}{P_0(\omega, t)} \right) \frac{\partial P_0(\omega, t)}{\partial \beta_k(\omega, t)} \\ = & \mathbb{E}_{\mathbf{x}} [D_n(\omega, t, \mathbf{x})] \left(1 - \frac{\mathbb{E}_{\mathbf{x}} [u'(W(\omega, t, \mathbf{x} + \sqrt{T-t}\sigma_n))]}{\mathbb{E}_{\mathbf{x}} [u'(W(\omega, t, \mathbf{x}))]} \right) \frac{\partial P_0(\omega, t)}{\partial \beta_k(\omega, t)} \end{aligned} \quad (5)$$

The Asset-Riskiness Effect

Given $\sqrt{T-t}\mathbf{x}$, the extent to which $d\beta_k(\omega, t)$ alters $P_n(\omega, t)$ by changing the marginal utility of terminal-period wealth depends on the future realization of the n th terminal dividend. Similarly, the extent to which $d\beta_k(\omega, t)$ alters $P_n(\omega, t)$ via a change in the n th terminal dividend depends on the future realization of the marginal utility of terminal-period wealth. Which is to say that changes in the realization $\beta_k(\omega, t)$ affect the equilibrium price of the n th risky security through changes in the correlation between the marginal utility of terminal-period wealth and the terminal dividend of the security. I will be referring to this as the *asset-riskiness effect* of $d\beta_k(\omega, t)$ on $P_n(\omega, t)$. It is depicted by the first term on the right-hand side of equation (1).

To understand the mechanics of this effect, it is instructive to consider a setting in which (i) the components of the Brownian process that determine the n th terminal dividend ($\beta_m(\omega, t)$ with $\sigma_{nm} \neq 0$) affect the terminal-period wealth only through this dividend and (ii) the k th Brownian component does not affect the dividend ($\sigma_{nk} = 0$). Formally, let

$$K_n = \{m \in \{1, \dots, K\} : \sigma_{nm} \neq 0\}$$

be the collection of those Brownian components that determine $D_n(\omega, T)$. Suppose also that $k \notin K_n$ and let $M = |K_n| < K$, $\mathbf{z} \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_M)$, $\mathbf{y} \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_{K-M})$, $\mathbf{x} = (\mathbf{z}, \mathbf{y})$, and consider

the terminal-period wealth specification

$$\begin{aligned} W(\omega, t, \mathbf{x}) &= \rho(\omega, t, \mathbf{y}) + \sum_{n' \notin K_n} D_{n'}(\omega, t, \mathbf{y}) + D_n(\omega, t, \mathbf{z}) \\ &= W_{-M}(\omega, t, \mathbf{y}) + D_n(\omega, t, \mathbf{z}) \end{aligned} \quad (6)$$

for some continuous function $\rho : \mathbb{R}^{K-M} \mapsto \mathbb{R}_+$.

In this case, $\frac{\partial W(\omega, t, \mathbf{x})}{\partial \beta_k(\omega, t)} = \frac{\partial W_{-M}(\omega, t, \mathbf{y})}{\partial \beta_k(\omega, t)}$ and the first term on the right-hand side of (1) can be written out as follows

$$\begin{aligned} & \text{Cov}_{\mathbf{x}} \left[u''(W(\omega, t, (\mathbf{z}, \mathbf{y}))) \frac{\partial W(\omega, t, \mathbf{y})}{\partial \beta_k(\omega, t)}, D_n(\omega, t, \mathbf{z}) \right] \\ &= \int_{\mathbb{R}^{K-M}} \left(\int_{\mathbb{R}^M} u''(W(\omega, t, (\mathbf{z}, \mathbf{y}))) D_n(\omega, t, \mathbf{z}) d\Phi(\mathbf{z}) - \int_{\mathbb{R}^M} u''(W(\omega, t, (\mathbf{z}, \mathbf{y}))) d\Phi(\mathbf{z}) \int_{\mathbb{R}^M} D_n(\omega, t, \mathbf{z}) d\Phi(\mathbf{z}) \right) \frac{\partial W_{-M}(\omega, t, \mathbf{y})}{\partial \beta_k(\omega, t)} d\Phi(\mathbf{y}) \\ &= \int_{\mathbb{R}^{K-M}} \text{Cov}_{\mathbf{z}} [u''(W(\omega, t, (\mathbf{z}, \mathbf{y}))), D_n(\omega, t, \mathbf{z})] \frac{\partial W_{-M}(\omega, t, \mathbf{y})}{\partial \beta_k(\omega, t)} d\Phi(\mathbf{y}) \end{aligned} \quad (7)$$

Conditional on the realization \mathbf{y} , the terminal-period wealth $W(\omega, t, \mathbf{x})$ is strictly comonotonic in \mathbf{z} with $D_n(\omega, t, \mathbf{z})$. Under non-increasing absolute risk aversion (NARA), so is $u''(W(\omega, t, \mathbf{x}))$ and the covariance within the integral above is strictly positive (see Appendix B).¹² Which means that the sign of the asset-riskiness effect of $d\beta_k(\omega, t)$ on $P_n(\omega, t)$ will be given by the sign of $\frac{\partial W_{-M}(\omega, t, \mathbf{y})}{\partial \beta_k(\omega, t)}$, as long as the latter sign remains unchanged on \mathbb{R}^{K-M} .

Recall, however, that the wealth effect of $d\beta_k(\omega, t)$ on $P_n(\omega, t)$ obtains always in the same direction as the wealth effect on $P_0(\omega, t)$. And $\frac{\partial P_0(\omega, t)}{\partial \beta_k(\omega, t)}$ is required by (2) to have the opposite sign of $\frac{\partial W_{-M}(\omega, t, \mathbf{y})}{\partial \beta_k(\omega, t)}$. In this setting, therefore, the asset-riskiness and wealth effects push $P_n(\omega, t)$ in opposite directions under NARA and the intuition why is straightforward. Let, for instance, $\frac{\partial W_{-M}(\omega, t, \mathbf{y})}{\partial \beta_k(\omega, t)} > 0 \forall \mathbf{y} \in \mathbb{R}^{K-M}$. An increase in $\beta_k(\omega, t)$ raises the \mathcal{F}_t -conditional terminal-period wealth, reducing (due to risk aversion) its marginal utility. Under NARA, though, the decrease in $u'(W(\omega, t, \mathbf{x}))$ is smaller when $D_n(\omega, t, \mathbf{z})$ is large and larger when it is small. That is, the increase in $\beta_k(\omega, t)$ makes the marginal utility of terminal-period wealth and the n th terminal dividend less negatively correlated. Equivalently (due to risk aversion), the terminal-period wealth is now less positively correlated with the n th terminal dividend. Which diminishes the agent's perceived "riskiness" of the n th security, inducing her to demand more of it and (in the face of fixed supply) raise its price in equilibrium.

¹²The coefficient of absolute risk-aversion is the function $r_A : \mathbb{R} \rightarrow \mathbb{R}_{++}$ defined by $r_A(\cdot) = -u''(\cdot)/u'(\cdot)$. It is non-increasing ($r'_A(\cdot) \geq 0$) only if $u'''(\cdot) \geq -u''(\cdot)r_A(\cdot) > 0$.

A concrete example of this type of equilibrium price change due to the asset-riskness effect is provided by Corollary 3.1.4. This assumes a setting in which and the terminal dividends of the n th and some other security, denoted by n' , vary with the m th and the k th Brownian motions, respectively, with the former Brownian component being the only source of stochastic variations in the n th dividend, $\sigma_n = \sigma_{jm}\mathbf{e}_m$.¹³ These two components, moreover, do not affect other components of the terminal-period wealth, $\frac{\partial \rho(\omega, T)}{\partial \beta_k(\omega, t)} = \frac{\partial \rho(\omega, T)}{\partial \beta_m(\omega, t)} = 0$. The corresponding terminal-period wealth specification is a special case of (6):

$$\begin{aligned} W(\omega, t, \mathbf{x}) &= \rho(\omega, t, \mathbf{x}_{-(k, m)}) + \sum_{i \in \{1, \dots, N\} \setminus \{n, n'\}} D_i(\omega, t, \mathbf{x}_{-(k, m)}) \\ &\quad + e^{\mu_{n'}T + \sigma_{n'k}(\beta_{n'k}(\omega, t) + \sqrt{T-t}x_k)} + e^{\mu_n T + \sigma_n(\beta_m(\omega, t) + \sqrt{T-t}x_m)} \\ &= W_{-(k, m)}(\omega, t, \mathbf{x}_{-(k, m)}) + D_{n'}(\omega, t, x_k) + D_n(\omega, t, x_m) \end{aligned} \quad (8)$$

As established by the corollary, in this case, under DARA, the relative equilibrium price of the n th security is increasing (decreasing) in the realization $\beta_k(\omega, t)$ if $\sigma_{n'k} > 0$ ($\sigma_{n'k} < 0$). And this obtains even though the wealth effect on the relative price has the same sign as the wealth effect on the price of the bond, negative (positive) if $\sigma_{n'k} > 0$ ($\sigma_{n'k} < 0$).¹⁴ Therefore, in this example, the monotonicity of $p_n(\omega, t)$ with respect to $\beta_k(\omega, t)$ is due to the fact that the asset riskness effect of $\beta_k(\omega, t)$ on $p_n(\omega, t)$ dominates the wealth effect.

Needless to say, once we allow for the n th terminal dividend to depend upon the k th Brownian motion ($\sigma_{nk} \neq 0$), the mechanics of the asset-riskiness effect become more complicated. Given a change $d\beta_k(\omega, t)$, the new level of terminal-period wealth will be $W(\omega, t, \mathbf{x}) + dW(\omega, t, \mathbf{x})$. The new covariance of the marginal utility of terminal-period wealth with the n th terminal dividend is given by

$$\begin{aligned} &\text{Cov}_{\mathbf{x}} \left[u'(W(\omega, t, \mathbf{x}) + dW(\omega, t, \mathbf{x})), e^{\mu_n T + \sigma_n^{\top}(\beta(\omega, t) + d\beta_k(\omega, t)\mathbf{e}_k + \sqrt{T-t}\mathbf{x})} \right] \\ &= e^{\sigma_{nk} d\beta_k(\omega, t)} \text{Cov}_{\mathbf{x}} [u'(W(\omega, t, \mathbf{x}) + dW(\omega, t, \mathbf{x})), D_n(\omega, t, \mathbf{x})] \end{aligned}$$

¹³As usual, for $K \in \mathbb{N}$ and $m \in \{1, \dots, K\}$, $\mathbf{e}_m \in \mathbb{R}^K$ denotes the vector with 1 at its m th entry and zeroes elsewhere. In addition, for $k' \in \{1, \dots, K\}$ with $k' < m$ and $\mathbf{x} \in \mathbb{R}^K$, \mathbf{x}_{-m} and $\mathbf{x}_{-(k', m)}$ denote, respectively, the vectors $(x_1, \dots, x_{m-1}, x_{m+1}, \dots, x_K)^{\top} \in \mathbb{R}^{K-1}$ and $(x_1, \dots, x_{k'-1}, x_{k'+1}, \dots, x_{m-1}, x_{m+1}, \dots, x_K)^{\top} \in \mathbb{R}^{K-2}$.

¹⁴Recall that the wealth effect of $d\beta_k(\omega, t)$ operates in the same direction on all absolute prices. To show, therefore, that it pulls also all relative prices in this direction, it is enough to show that it drives the relative price of the n th risky security in the direction in which it pushes the price of the bond. And for this it suffices that the expression in the brackets on the right-hand side of (5) is positive. Which follows immediately by risk aversion ($u''_2(\cdot) < 0$). Because, for $\sigma_n = \sigma_{nm}\mathbf{e}_m$, (??) gives $W(\omega, t, \mathbf{x} + \sqrt{T-t}\sigma_n) = W(\omega, t, \mathbf{x}) + D_n(\omega, t, x_m + \sqrt{T-t}\sigma_{jm}) - D_n(\omega, t, x_m)$ with $D_n(\omega, t, x_m + \sqrt{T-t}\sigma_{jm}) = e^{\mu_n T + \sigma_{jm}(\beta_m(\omega, t) + \sqrt{T-t}(y_m + \sqrt{T-t}\sigma_{jm}))} = \left(e^{(T-t)\sigma_{jm}^2} - 1 \right) D_n(\omega, t, x_m)$.

What happens to the perceived “riskness” of the n th security is determined now, not only by what happens to the correlation between the marginal utility at the new terminal-period wealth and the n th dividend (i.e. by comparing the covariance on the right-hand side above with the initial $\text{Cov}_{\mathbf{x}}[u'(W(\omega, t, \mathbf{x})), D_n(\omega, t, \mathbf{x})]$), but also by the term $e^{\sigma_{nk}d\beta_k(\omega, t)}$.

For instance, suppose once again that $W(\omega, t, \mathbf{x})$ and $D_n(\omega, t, \mathbf{x})$ are strictly comonotonic in \mathbf{x} so that $u'(W(\omega, t, \mathbf{x}))$ is strictly countermonotonic and, thus, negatively correlated with $D_n(\omega, t, \mathbf{x})$. Let also $\sigma_{nk}d\beta_k(\omega, t) > 0$ so that $e^{\sigma_{nk}d\beta_k(\omega, t)} > 1$. Even if, as in the preceding example, the change in terminal-period wealth renders its marginal utility less negatively correlated with the n th terminal dividend, the increase in the dividend’s drift might be sufficient to make now their new covariance more negative overall. As opposed to the preceding example, the perceived “riskness” of the n th security would increase with $\beta_k(\omega, t)$, exerting a downward pressure on its equilibrium relative price.

The direction and importance of the asset-riskiness effect for the dynamics of relative prices depends also on the agent’s utility function; namely, her risk-aversion. Consider, for instance, the following setting. The agent exhibits CARA and the m th Brownian motion affects both the n th and n' th terminal dividends. The former dividend is independent of any other Brownian component ($\sigma_n = \sigma_{nm}\mathbf{e}_m$). The latter varies also with but only with the k th Brownian motion ($\sigma_{n'} = \sigma_{n'm}\mathbf{e}_m + \sigma_{n'k}\mathbf{e}_k$), which, in turn, affects no other component of the terminal-period wealth, $\frac{\partial \rho(\omega, T)}{\partial \beta_k(\omega, t)} = 0$. The corresponding terminal-period wealth specification is given by

$$\begin{aligned} W(\omega, t, \mathbf{x}) &= \rho(\omega, t, \mathbf{x}_{-k}) + \sum_{i \in \{1, \dots, N\} \setminus \{n, n'\}} D_i(\omega, t, \mathbf{x}_{-k}) + e^{\mu_n T + \sigma_n(\beta_m(\omega, t) + \sqrt{T-t}x_m)} \\ &\quad + e^{\mu_{n'} T + \sigma_{n'm}\beta_m(\omega, t) + \sigma_{n'k}\beta_k(\omega, t) + \sqrt{T-t}(\sigma_{n'm}x_m + \sigma_{n'k}x_k)} \\ &= W_{-k}(\omega, t, \mathbf{x}_{-k}) + D_n(\omega, t, x_m) + D_{n'}(\omega, t, (x_k, x_m)) \end{aligned} \quad (9)$$

Corollary 3.1.3 dictates that, as long as $\sigma_{n'm}\sigma_{n'm} > 0$, a rise in $\beta_k(\omega, t)$ increases (decreases) here the relative price of the n th security if $\sigma_{n'k} < 0$ ($\sigma_{n'k} > 0$). To analyze this result in terms of the asset-riskness and wealth effects, we need to be able to determine the direction of the latter. It is easy to do so by restricting attention to the special case of (9) in which the m th Brownian motion affects no other component of the terminal-period wealth but the two dividends, $\frac{\partial \rho(\omega, T)}{\partial \beta_m(\omega, t)} = 0$ and $\sigma_{im} = 0 \forall i \in \{1, \dots, N\} \setminus \{n, n'\}$. The specification in question

$$W(\omega, t, \mathbf{x}) = W_{-(k,m)}(\omega, t, \mathbf{x}_{-(k,m)}) + D_n(\omega, t, x_m) + D_{n'}(\omega, t, (x_k, x_m))$$

gives

$$W\left(\omega, t, \mathbf{x} + \sqrt{T-t}\sigma_n\right) = W_{-(k,m)}\left(\omega, t, \mathbf{x}_{-(k,m)}\right) + \left(e^{(T-t)\sigma_{nm}^2} - 1\right) D_n\left(\omega, t, x_m\right) \\ + \left(e^{(T-t)\sigma_{n'm}\sigma_{nm}} - 1\right) D_{n'}\left(\omega, t, (x_m, x_k)\right)$$

If $\sigma_{nm}\sigma_{n'm} > 0$, $W\left(\omega, t, \mathbf{x} + \sqrt{T-t}\sigma_n\right) > W\left(\omega, t, \mathbf{x}\right)$ and, by (5), the wealth effect of $d\beta_k\left(\omega, t\right)$ on the n th relative price must operate in the same direction as it does on the price of the bond. But $\sigma_{n'k} \frac{\partial W(\omega, t, \mathbf{x})}{\partial \beta_k(\omega, t)} > 0$ here, so that - recall (2) - the wealth effect increases (decreases) the bond price with $\beta_k\left(\omega, t\right)$ if $\sigma_{n'k} < 0$ ($\sigma_{n'k} > 0$). Contrary to the DARA example, therefore, a change in $\beta_k\left(\omega, t\right)$ alters the n th relative price in the direction of the wealth effect, irrespectively of the asset-riskness effect.

The Combined Effect

Recall that the relative equilibrium price of the n th risky security can be written as

$$p_n\left(\omega, t\right) = \mathbb{E}_{\mathbf{x}}\left[D_n\left(\omega, t, \mathbf{x}\right)\right] + \frac{\text{Cov}_{\mathbf{x}}\left[u'\left(W\left(\omega, t, \mathbf{x}\right)\right), D_n\left(\omega, t, \mathbf{x}\right)\right]}{P_0\left(\omega, t\right)} \quad (10)$$

Its dynamics, with respect to changes in the current information $\beta_k\left(\omega, t\right)$, are determined by two factors, the own-dividend effect and the asset-riskness relative to the price of the bond. Since, however,

$$\left(\frac{1}{p_n\left(\omega, t\right)}\right) \frac{\partial p_n\left(\omega, t\right)}{\partial \beta_k\left(\omega, t\right)} = \left(\frac{1}{P_n\left(\omega, t\right)}\right) \frac{\partial P_n\left(\omega, t\right)}{\partial \beta_k\left(\omega, t\right)} - \left(\frac{1}{P_0\left(\omega, t\right)}\right) \frac{\partial P_0\left(\omega, t\right)}{\partial \beta_k\left(\omega, t\right)} \quad (11)$$

these dynamics are determined by the difference between the relative changes in the absolute prices, $P_n\left(\omega, t\right)$ and $P_0\left(\omega, t\right)$. And, in general, this is complex enough to be rather impossible to predict using only economic intuition. First, the wealth effects on the two absolute prices, by pushing them in the same direction, pull $p_n\left(\omega, t\right)$ in opposite directions. Second, the own-dividend effect on $P_n\left(\omega, t\right)$ pushes it always in the opposite direction than its wealth effect. Finally, as shown by the preceding examples, if $u\left(\cdot\right)$ exhibits NARA, the asset-riskiness effect may pull $p_n\left(\omega, t\right)$ in the opposite direction than the wealth effect.

Theorem 2.1 addresses these issues unequivocally but for the dynamics of the relative price of the security with respect to the current realization of the entire Brownian vector, $\beta\left(\omega, t\right)$. It ensures that the inner product of the n th row of the dispersion matrix Σ with the gradient vector of the n th relative price, $\nabla_{\beta(\omega, t)} p_n\left(\omega, t\right)$, is strictly positive as long as the terminal dividend of the security is stochastic, in the sense in which the uncertainty is

captured in this model.¹⁵

The intuition behind this result is straightforward when the terminal dividend is correlated with only the m th Brownian motion and this relation is exclusive: $\sigma_n = \sigma_{nm}\mathbf{e}_m$, $\frac{\partial p(\omega, T)}{\partial \beta_m(\omega, t)} = 0$, and $\sigma_{n'm} = 0 \forall n' \in \{1, \dots, N\} \setminus \{n\}$. Formally, the corresponding terminal-wealth specification is that in (6) for $M = 1$. In this setting, let $\beta_m(\omega, t)$ change by $d\beta_m(\omega, t)$. For any terminal realization x_m , the terminal-period wealth changes now only through the n th dividend, whose new value is

$$\begin{aligned} D_n(\beta_m(\omega, t) + d\beta_m(\omega, t), x_m) &= e^{\mu_n T + \sigma_{jm}(\beta_m(\omega, t) + d\beta_m(\omega, t) + \sqrt{T-t}x_m)} \\ &= e^{\sigma_{jm}d\beta_m(\omega, t)} D_n(\omega, t, x_m) \end{aligned}$$

Since the agent is everywhere non-satiated ($u'(\cdot) > 0$) and any other component of her terminal-period wealth remains unaffected by $d\beta_m(\omega, t)$, her preferences for the n th risky asset change in the direction of First-order Stochastic Dominance (FSD).

Suppose, specifically, that $\beta_m(\omega, t)$ increases (decreases). If $\sigma_{nm} > 0$, the new terminal dividend dominates (is dominated by) the old in the sense of FSD. The agent is now more (less) willing to hold the stock and, facing its fixed supply, pushes up its absolute price. Moreover, the wealth effect on the price of the bond is negative (positive). Clearly, the relative price of the security increases (decreases). If $\sigma_{jm} < 0$, on the other hand, the new terminal dividend is dominated by (dominates) the old in terms of FSD whereas the wealth effect on $P_0(\omega, t)$ is positive (negative). In either case, therefore, $\sigma_{nm} \frac{\partial p_n(\omega, t)}{\partial \beta_m(\omega, t)} > 0$.¹⁶

In more complex settings, the theorem can be viewed as generalizing this argument to the dynamics of the relative price with respect to the current realization of entire Brownian vector. Its proof (see Appendix C) uses straightforward mathematical apparatus but is quite subtle in its reasoning, especially with respect to its last (and most crucial) step. It attests to the complexity of the dynamics of the equilibrium relation between the relative prices and

¹⁵For $\sigma_n = \mathbf{0}$, we get $p_n(\omega, t) = e^{\mu_n T}$. The relative price is constant, independent of the realizations of the Brownian motions: $\frac{\partial p_n(\omega, t)}{\partial \beta_k(\omega, t)} = 0 \forall k \in \{1, \dots, K\}$. Consider the typical Brownian dimension. Since $\sigma_{nk} = 0$, there is no own-dividend effect on $P_n(\omega, t)$. Since all other factor loadings of the n th terminal dividend are also zero, the dividend is independent of the subsequent path $\{\beta(\omega, \tau) : \tau \in (t, T]\}$ of the Brownian process and, consequently, of the terminal-period wealth. Clearly, a change in $\beta_k(\omega, t)$ induces no asset-riskiness effect on $P_n(\omega, t)$ while its wealth effects on $P_n(\omega, t)$ and $P_0(\omega, t)$ cancel each other out.

¹⁶Put differently, when $\sigma_{nm} > 0$ ($\sigma_{nm} < 0$), going from the old to the new terminal dividend is in the opposite (same) direction as Proposition 1 in Gollier [21], the factor being $e^{\sigma_{jm}d\beta_m(\omega, t)}$. For any risk-averse individual, $d\beta_m(\omega, t)$ increases (reduces) the optimal demand and, consequently, the equilibrium relative price of the n th risky security. Of course, Gollier [21] studies probability distributions whose supports are closed intervals but his results are straightforward to generalize in my context (see Lemma A.1 in Appendix A).

the current realization of the underlying stochastic process. In the next section, I restrict attention to situations in which there is sufficient structure in the economic primitives for more precise conclusions to be made.

3 The Dynamics of Relative Prices

Let me begin with the trivial yet important case in which there is only risky security and a single source of uncertainty ($N = K = 1$). As $\beta(\omega, t)$ and Σ are now scalars (the latter denoted by s for notational convenience), we have

$$D(\mathcal{I}(\omega, t), x) = e^{\mu T + s(\beta(\omega, t) + \sqrt{T-t}x)} \quad s \frac{\partial p_n(\omega, t)}{\partial \beta(\omega, t)} > 0$$

regarding the terminal dividend of the risky security and the claim of Theorem 2.1. The equilibrium relative price is always monotone with respect to the current Brownian realization. More precisely, when $\beta(\omega, t)$ increases, $p(\omega, t)$ increases (decreases) if the dividend is positively (negatively) correlated with the Brownian motion. As a consequence, the path of the equilibrium relative price process identifies uniquely the path on which the underlying uncertainty is being resolved.

When $N = K = 1$, therefore, Theorem 2.1 determines unequivocally the combined effect of the potentially contradicting forces highlighted by the analysis of the previous section. To illustrate, let the agent exhibit DARA and her terminal wealth be increasing in the current realization of the underlying stochastic process, $\frac{\partial W(t, \mathcal{I}(\omega, T))}{\partial \beta(\omega, t)} > 0$ (which would be the case, for example, if $s > 0$ and $\frac{\partial e(t, \mathcal{I}(\omega, T))}{\partial \beta(\omega, t)} \geq 0$). If $s > 0$, an increase in $\beta(\omega, t)$ raises the \mathcal{F}_t -conditional terminal dividend of the security, pushing $P(\omega, t)$ upwards through the own-dividend effect. It increases, though, also the terminal wealth, exerting negative wealth effects on both $P_0(\omega, t)$ and $P(\omega, t)$. And to make matters more complicated, as pointed out in the previous section, the asset-riskiness effect on $P(\omega, t)$ may go in either direction. Nevertheless, the combined effect on the price of the risky security is such that, even though the price of the bond necessarily falls, that of the stock either increases or decreases by less in percentage terms.

By contrast, with more than one sources of uncertainty ($K > 1$), the implications of the theorem are quite subtle even when $N = 1$. To facilitate the analysis, it is instructive to distinguish between whether the terminal dividend of the risky security under examination depends upon one or more components of the underlying stochastic process.

3.1 One-dimensional Dividend Uncertainty

Given a vector Brownian process, I will consider first the situation in which the terminal dividend of the n th risky security varies with the terminal-period realization of only one of its components, $\sigma_n = \sigma_{nm}\mathbf{e}_m$ for some $m \in \{1, \dots, K\}$. In this case,

$$D_n(\mathcal{I}(\omega, t), \mathbf{x}) = e^{\mu_n T + \sigma_{nm}(\beta_m(\omega, t) + \sqrt{T-t}x_m)} \quad (12)$$

and, as follows immediately by Theorem 2.1, the equilibrium relative price process of the n th security will be monotone in $\beta_m(\omega, t)$. More precisely, we have the following result.

Claim 3.1.1 *Let the terminal dividend of the n th risky security be given by (12). Then,*

$$\sigma_{nm} \frac{\partial p_n(\omega, t)}{\partial \beta_m(\omega, t)} > 0$$

Holding constant all other sources of uncertainty but the m th Brownian component, the path of the n th equilibrium relative price identifies uniquely the path $\{\beta_m(\omega, \tau) : \tau \in (t, T]\}$ in which the associated uncertainty gets resolved. Of course, this is true for every risky security in the model when the underlying Brownian dispersion matrix is diagonal ($N = K$ and $\Sigma = [\sigma_{11}\mathbf{e}_1, \dots, \sigma_{KK}\mathbf{e}_K]$).

Corollary 3.1.1 *Σ diagonal implies*

$$\sigma_{nn} \frac{\partial p_n(\omega, t)}{\partial \beta_n(\omega, t)} > 0 \quad \forall n = 1, \dots, K$$

Having identified the relation between the relative price of the n th security and the realization $\beta_m(\omega, t)$, let us turn to examining the cross-correlations, between this relative price and the current realization of a Brownian component that does not affect the n th terminal dividend, $\beta_k(\omega, t)$ with $k \neq m$. As Theorem 2.1 is no longer enough to facilitate by itself the analysis of these comparative statics, their investigation will be my focus for the remaining of this section. And I will show that, apart from quite special cases, the n th equilibrium relative price *does* vary with the realization $\beta_k(\omega, t)$. How it does depends on (i) the way in which the terminal wealth varies with the terminal-date realization of the Brownian process and (ii) the functional form (the risk-attitude in particular) of the agent's terminal utility.

Clearly, as $\sigma_{nk} = 0$, a change in the current realization $\beta_k(\omega, t)$ does not affect the \mathcal{F}_t -conditional drift, $\mu_n T + \sigma_n^\top \beta(\omega, t)$, of the stochastic process that drives the terminal dividend

of the n th security. Which is to say that there is no own-dividend effect on its equilibrium price, only wealth and asset-riskiness effects. To illustrate the importance of the latter for the richness of the relative price dynamics, I will consider first the case of an agent with CARA terminal utility: $u(c) = \gamma e^{\alpha c}$ with $\gamma, \alpha < 0$. For, as is well known, under CARA, changes in wealth that do not affect the risk premium of an asset should leave its relative equilibrium price unchanged. If the change in $\beta_k(\omega, t)$ results in such a wealth change, therefore, the asset-riskiness effect on the absolute price $P_n(\omega, t)$ should exactly cancel out the wealth effect on the relative price $p_n(\omega, t)$.

A sufficient condition for this to happen is that the k th and m th Brownian components affect the agent's terminal wealth through independent channels. This obtains under either of two terminal wealth functional specifications. In the first, the k th Brownian component affects the terminal dividend of only one of the remaining $N - 1$ risky securities and in an exclusive way: the dividend in question varies only with the k th Brownian component. And, similarly, if the k th Brownian component affects the terminal-period endowment process, it does so exclusively: it can be correlated with only one component of the endowment process and, in this case, the endowment component in question varies with only the k th Brownian component. Formally, let $\sigma_{n'} = \sigma_{n'k} \mathbf{e}_k$ and $N_k = \{n'\}$ for some $n' \in \{1, \dots, N\} \setminus \{n\}$. Suppose also that the terminal-period endowment process is given by

$$e(\mathcal{I}(\omega, T)) = e_1(\beta_{-k}(\omega, T)) + e_2(\beta_k(\omega, T))$$

for some continuous functions $e_1 : \mathbb{R}^{K-1} \mapsto \mathbb{R}_+$ and $e_2 : \mathbb{R} \mapsto \mathbb{R}_+$. The terminal wealth can now be expressed as

$$\begin{aligned} W(\mathcal{I}(\omega, t), \mathbf{x}) &= e_1\left(\left(\beta_{-k}(\omega, t) + \sqrt{T-t} \mathbf{x}_{-k}\right)\right) + e_2\left(\beta_k(\omega, t) + \sqrt{T-t} x_k\right) \\ &+ \sum_{i \in \{1, \dots, N\} \setminus \{n, n'\}} D_i(\omega, t, \mathbf{x}) + D_n(\omega, t, x_m) + D_{n'}(\omega, t, x_k) \end{aligned} \quad (13)$$

In the second terminal wealth specification, it is the m th Brownian component that affects the terminal wealth separately from the remaining $K - 1$ components of the Brownian process. It does so via exclusive relations with at most two components of the terminal wealth: for sure through the n th terminal dividend and possibly through some component of the terminal-period endowment process. To give a formal description again, for some continuous functions $e_1 : \mathbb{R}^{K-1} \mapsto \mathbb{R}_+$ and $e_2 : \mathbb{R} \mapsto \mathbb{R}_+$, let

$$e(\mathcal{I}(\omega, T)) = e_1(\beta_{-m}(\omega, T)) + e_2(\beta_m(\omega, T))$$

and

$$\begin{aligned}
W(\mathcal{I}(\omega, t), \mathbf{x}) &= e_1 \left(\beta_{-m}(\omega, t) + \sqrt{T-t} \mathbf{x}_{-m} \right) + e_2 \left(\beta_m(\omega, t) + \sqrt{T-t} x_m \right) \\
&+ \sum_{i \in \{1, \dots, N\} \setminus \{n\}} D_i(\omega, t, \mathbf{x}_m) + D_n(\omega, t, x_m)
\end{aligned} \tag{14}$$

As I have already asserted and establish formally in the following claim, under either of these terminal-period wealth specifications, changes in the k th component of the Brownian process leave the relative equilibrium price of the n th risky security unaffected.¹⁷

Proposition 3.1.1 *Suppose that the following conditions apply.*

- (i) *The agent's terminal utility exhibits CARA.*
- (ii) *The terminal dividend of the n th risky security is given by (12)*
- (iii) *For $k \in \{1, \dots, K\} \setminus \{m\}$, the terminal wealth process is specified as in (13) or (14)*

Then,

$$\frac{\partial p_n(\omega, t)}{\partial \beta_k(\omega, t)} = 0$$

An important special case of the specifications (13)-(14) obtains when (i) the dispersion matrix Σ is diagonal and (ii) the terminal-period endowment process is separable along the K dimensions of the Brownian vector, i.e.

$$e(\mathcal{I}(\omega, T)) = \sum_{i=1}^K e_i \left(\beta_i(\omega, t) + \sqrt{T-t} x_i \right)$$

for some continuous functions $e_i : \mathbb{R} \rightarrow \mathbb{R}_+$. The corresponding terminal wealth specification now is

$$W(\mathcal{I}(\omega, t), \mathbf{x}) = \sum_{i=1}^K e_i \left(\beta_i(\omega, t) + \sqrt{T-t} x_i \right) + \sum_{i=1}^K D_i(\omega, t, x_i) \tag{15}$$

and Proposition 3.1.1 requires that $\frac{\partial p_n(\omega, t)}{\partial \beta_k(\omega, t)} = 0 \forall k \in \{1, \dots, K\} \setminus \{n\}$. In this case, the dispersion matrix of the equilibrium relative prices

$$J(\mathcal{I}(\omega, t)) = \left[\frac{\partial p_n(\omega, t)}{\partial \beta_k(\omega, t)} \right]_{(n,k) \in \{1, \dots, K\} \times \{1, \dots, K\}}$$

¹⁷In fact, one can show explicitly that, under either of the three terminal wealth specifications (13)-(15), $\beta_k(\omega, t)$ is not a functional argument of $p_n(\omega, t)$. See equations (27)-(29) in Appendix C.

is diagonal. As, in addition, none of its diagonal elements can be zero (Corollary 3.1.1), it is nonsingular. Which in turn implies that the securities market is dynamically complete.¹⁸

Corollary 3.1.2 *Suppose that the following conditions are satisfied.*

- (i) *The agent's terminal utility exhibits CARA.*
- (ii) *The dispersion matrix Σ is diagonal.*
- (iii) *The terminal wealth process specification is given by (15).*

Then $J(\mathcal{I}(\omega, t))$ is diagonal, its n th diagonal element having the sign of σ_{nn} . The securities market is dynamically complete.

3.1.1 Contagion under CARA

As changes in the k th Brownian motion leave the n th terminal dividend unaffected ($\sigma_{nk} = 0$), Proposition 3.1.1 is in line with the intuition that, under CARA, changes in wealth that are independent of an asset's dividend should not matter for its equilibrium relative price. Yet, even though this a rather commonly held view about relative price dynamics within a CARA environment, the fact that $p_n(\mathcal{I}(\omega, t))$ does not respond to changes in $\beta_k(\omega, t)$ is not only due to $\sigma_{nk} = 0$.

It also depends, and fundamentally so, upon the separability of the channels through which the k th and m th (the one that does affect the n th dividend) Brownian motions operate on the terminal wealth in (13) and (14). For as the example I construct below shows (Proposition 3.1.2), once we allow the two Brownian components to influence the agent's terminal wealth through a common element, the relative equilibrium price of the security will no longer be unresponsive to changes in $\beta_k(\omega, t)$, even when the CARA and $\sigma_{nk} = 0$ assumptions are maintained. In fact, within the realm of Proposition 3.1.2, $p_n(\omega, t)$ varies monotonically with $\beta_k(\omega, t)$.

To see what is so special about the underlying separability in the specifications (13) and (14), it is instructive to consider equation (24) in Appendix C, which gives the rates of change of the equilibrium absolute prices of the n th risky security and the bond with respect to the

¹⁸Notice that, in the light of Theorem 4.1, dynamic completeness implies that the underlying dispersion matrix Σ must be invertible. Which is indeed the case. In fact, under the specification in (15), Σ is necessarily diagonal.

current realization of the k th Brownian motion. For $\sigma_{nk} = 0$, since $\mathbb{E}[x_k] = 0$, it reads

$$\begin{aligned} \frac{\partial P_n(\omega, t)}{\partial \beta_k(\omega, t)} &= \frac{1}{T-t} \mathbb{E}_{\mathbf{x}} \left[\sqrt{T-t} x_k u'(W(\omega, t, \mathbf{x})) D_n(\omega, t, \mathbf{x}) \right] \\ &= \frac{1}{T-t} \text{Cov}_{\mathbf{x}} \left[\sqrt{T-t} x_k, u'(W(\omega, t, \mathbf{x})) D_n(\omega, t, \mathbf{x}) \right] \\ \frac{\partial P_0(\omega, t)}{\partial \beta_k(\omega, t)} &= \frac{1}{T-t} \mathbb{E}_{\mathbf{x}} \left[\sqrt{T-t} x_k u'(W(\omega, t, \mathbf{x})) \right] = \frac{1}{T-t} \text{Cov}_{\mathbf{x}} \left[\sqrt{T-t} x_k, u'(W(\omega, t, \mathbf{x})) \right] \end{aligned}$$

Either equation is in terms of the \mathcal{F}_t -conditional covariance between the marginal utility of terminal wealth (and, thus, consumption) that is derived from holding an extra unit of the security and the future realizations of the k th Brownian motion, $\beta(\omega, T) - \beta(\omega, t)$. It is trivial to check that, when the agent's utility exhibits CARA and her terminal wealth is given by either of the specifications (13) and (14), $\frac{\partial P_n(\omega, t)}{\partial \beta_k(\omega, t)} = \frac{\partial P_0(\omega, t)}{\partial \beta_k(\omega, t)}$. But then condition (11) precludes any changes in the relative price of the security. In this case, a change in the realization $\beta_k(\omega, t)$ induces a percentage change in the covariance of the marginal utility of terminal wealth with the n th terminal dividend which is exactly equal to the percentage change it induces in the price of the bond. As a consequence, the covariance in question remains unchanged when measured in units of the bond, which means in turn that the second term on the right-hand side of (10) remains unaltered. And, as the expected terminal dividend cannot vary with $\beta_k(\omega, t)$, so does the relative price itself.

Yet, this is not generically the case. The following result refers to a situation in which the agent's utility exhibits CARA but the terminal wealth specification is such that the asset-riskiness and wealth effects of the realization $\beta_k(\omega, t)$ on the n th relative price do not cancel each other out. In fact, their interaction result in the relative price varying monotonically with the realization. The specification in question can be expressed as

$$W(\omega, t, \mathbf{x}) = e(\omega, t, \mathbf{x}_{-k}) + \sum_{i \notin N_k} D_i(\omega, t, \mathbf{x}_{-k}) + \sum_{l \in N_k} D_l(\omega, t, \mathbf{x}) + D_n(\omega, t, x_m) \quad (16)$$

where

$$N_k = \{i \in \{1, \dots, N\} : \sigma_{ik} \neq 0\}$$

denotes the collection of those risky securities whose terminal dividends do vary with the k th component of the Brownian vector (and, obviously, here $n \notin N_k$).

Proposition 3.1.2 *Let the following conditions apply.*

- (i) *The agent's terminal utility exhibits CARA.*

(ii) The terminal dividend of the n th risky security is given by (12)

(iii) For some $k \neq m$, the k th Brownian motion affects only dividends: $\frac{\partial e(\mathcal{I}(\omega, T))}{\partial \beta_k(\omega, T)} = 0$.

(iv) For the terminal dividends which are correlated with both the m th and k th Brownian motions, their respective factor loadings are such that $\prod_{n' \in N_m \cap N_k} \sigma_{n'm} \sigma_{n'k} > 0$.

Then,

$$\sigma_{nm} \sigma_{n'm} \sigma_{n'k} \frac{\partial p_n(\omega, t)}{\partial \beta_k(\omega, t)} < 0 \quad \forall n' \in N_m \cap N_k$$

As specific examples of the situation depicted by the proposition, consider the three dispersion matrices below. Each refers to an economy in which there are more than one risky securities and a bond while the underlying uncertainty is described by a three-dimensional Brownian motion. The first Brownian component represents macroeconomic uncertainty - it affects all risky assets (albeit with possibly different degrees of sensitivity) - and the first security is a “bet” exclusively on this macroeconomic uncertainty. The preceding result applies for the relative price of this exclusive security regarding changes in the realization of one of the other Brownian motion ($k \geq 2$) as long as the agent’s terminal-period endowment does not depend upon it.

$$\Sigma_1 = \begin{pmatrix} \sigma_{11} & 0 & 0 \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \end{pmatrix} \quad \sigma_{11} \sigma_{21} \sigma_{2k} \frac{\partial p_1(\omega, t)}{\partial \beta_k(\omega, t)} < 0 \quad k = 2, 3$$

$$\Sigma_2 = \begin{pmatrix} \sigma_{11} & 0 & 0 \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \quad \begin{aligned} \sigma_{11} \sigma_{n'1} \sigma_{n'2} \frac{\partial p_1(\omega, t)}{\partial \beta_2(\omega, t)} < 0 \quad n' = 2, 3 \\ \sigma_{11} \sigma_{n'1} \sigma_{n'3} \frac{\partial p_1(\omega, t)}{\partial \beta_3(\omega, t)} < 0 \quad n' = 2, 3 \end{aligned}$$

$$\Sigma_3 = \begin{pmatrix} \sigma_{11} & 0 & 0 \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & 0 & \sigma_{33} \end{pmatrix} \quad \begin{aligned} \sigma_{11} \sigma_{21} \sigma_{22} \frac{\partial p_1(\omega, t)}{\partial \beta_2(\omega, t)} < 0 \\ \sigma_{11} \sigma_{n'1} \sigma_{n'3} \frac{\partial p_1(\omega, t)}{\partial \beta_3(\omega, t)} < 0 \quad n' = 2, 3 \end{aligned}$$

Regarding condition (iii) of the proposition, in the first example it reads $\sigma_{21} \sigma_{2k} > 0$ for $k = 2, 3$. In the other two, it becomes $\sigma_{21} \sigma_{2k} \sigma_{31} \sigma_{3k} > 0$ for $k = 2, 3$ in the second economy and for $k = 3$ in the third (there, when $k = 2$, $N_1 \cap N_2$ is a singleton). Observe also that, in every case, $\sigma_{11} \frac{\partial p_1(\omega, t)}{\partial \beta_1(\omega, t)} > 0$ (Theorem 2.1) and, thus, we can sign the entire first row of the dispersion matrix of the relative price process.

As exemplified by the $k = 2$ case in the example of the matrix Σ_2 , the specification in (16) nests that in (9). Under the latter, the k th Brownian component affects only one terminal

dividend, which is also the only terminal wealth element that is correlated with both this and the m th Brownian motion. In this case, condition (iii) of Proposition 3.1.2 becomes redundant (the set $N_m \cap N_k$ is a singleton) and the claim can be stated as follows.

Corollary 3.1.3 *Let the following conditions apply.*

- (i) *The agent's terminal utility exhibits CARA.*
- (ii) *The terminal dividend of the n th risky security is given by (12)*
- (iii) *For some $k \neq m$, the k th Brownian motion affects no element of the agent's terminal wealth but one terminal dividend: $\frac{\partial e(\mathcal{I}(\omega, T))}{\partial \beta_k(\omega, T)} = 0$ and $N_k = \{n'\}$ for some $n' \in \{1, \dots, N\} \setminus \{n\}$.*
- (iv) *The terminal dividend of the n' security varies only with the k th and m th Brownian motions: $\sigma_{n'} = \sigma_{n'm} \mathbf{e}_m + \sigma_{n'k} \mathbf{e}_k$.*

Then,

$$\sigma_{nm} \sigma_{n'm} \sigma_{n'k} \frac{\partial p_n(\omega, t)}{\partial \beta_k(\omega, t)} < 0$$

The two dispersion matrices below are both compatible with the conditions of the corollary. As before, each refers to an economy in which the first component of the underlying stochastic process affects all risky assets and, possibly, also the terminal-period endowment process. The first asset is a “bet” exclusively on this source of macroeconomic uncertainty. In either case, the corollary applies for $k > 1$ as long as the k th Brownian motion is not correlated with the agent's terminal-period endowment process.

$$\Sigma_4 = \begin{pmatrix} \sigma_{11} & 0 \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \quad \sigma_{11} \sigma_{21} \sigma_{22} \frac{\partial p_1(\omega, t)}{\partial \beta_2(\omega, t)} < 0$$

$$\Sigma_5 = \begin{pmatrix} \sigma_{11} & 0 & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ \sigma_{31} & 0 & \sigma_{33} \end{pmatrix} \quad \sigma_{11} \sigma_{k1} \sigma_{kk} \frac{\partial p_1(\omega, t)}{\partial \beta_k(\omega, t)} < 0 \quad k = 2, 3$$

At first glance, these results might seem puzzling for they contradict the rather commonly held view that, under CARA, changes in wealth that are independent of an asset's payoff should not matter for its equilibrium relative price. In my view, the erroneously crude intuition behind this assertion stems from the multitude of examples in the financial economics literature that take the agent's wealth to be linearly-dependent upon asset payoffs. Although

rendering our work-horse discrete-time models analytically tractable and elegant, the linearity assumption obscures our grasp of the interaction between the asset-riskiness and wealth effects on the relative equilibrium price. For it forces this interaction to amount to nothing. And this is true irrespectively of the correlations between the various other elements of the agent's wealth.

To see why this happens, let $W(\mathcal{I}(\omega, T))$ be linear on the k th component of the Brownian vector: $\frac{\partial W(\omega, t, \mathbf{x})}{\partial \beta_k(\omega, t)} = \lambda_k$ for some $\lambda_k \in \mathbb{R}$ and for all $\mathbf{x} \in \mathbb{R}^K$. From (7), the asset-riskiness effect on the relative equilibrium price is now

$$\begin{aligned} \frac{\lambda_k}{P_0(\omega, t)} \text{Cov}_{\mathbf{x}} [u''(W(\omega, t, \mathbf{x})), D_n(\omega, t, \mathbf{x})] &= \frac{\alpha \lambda_k}{P_0(\omega, t)} \text{Cov}_{\mathbf{x}} [u'(W(\omega, t, \mathbf{x})), D_n(\omega, t, \mathbf{x})] \\ &= \alpha \lambda_k (p_n(\omega, t) - \mathbb{E}_{\mathbf{x}} [D_n(\omega, t, \mathbf{x})]) \end{aligned}$$

the first equality above following from CARA. But this is exactly the opposite of the wealth effect which is given by

$$\begin{aligned} &\frac{1}{P_0(\omega, t)} (\mathbb{E}_{\mathbf{x}} [D_n(\omega, t, \mathbf{x})] - p_n(\omega, t)) \frac{\partial P_0(\omega, t)}{\partial \beta_k(\omega, t)} \\ &= \frac{1}{P_0(\omega, t)} (\mathbb{E}_{\mathbf{x}} [D_n(\omega, t, \mathbf{x})] - p_n(\omega, t)) \mathbb{E}_{\mathbf{x}} \left[u''(W(\omega, t, \mathbf{x})) \frac{\partial W(\omega, t, \mathbf{x})}{\partial \beta_k(\omega, t)} \right] \\ &= \alpha \lambda_k (\mathbb{E}_{\mathbf{x}} [D_n(\omega, t, \mathbf{x})] - p_n(\omega, t)) \end{aligned}$$

Of course, given that the \mathcal{F}_t -conditional future realizations $\beta_k(\omega, T) - \beta_k(\omega, t)$ are normally-distributed here, the linearity assumption requires unlimited liability, an unrealistically strong condition as it implies that the agent may lose more than everything with positive probability. This is a well-known drawback. What my investigation of the CARA case shows is that, to make matters worse, the linearity assumption is restrictive also in a theoretical sense. When the representative agent exhibits CARA, it conditions the asset-riskiness and wealth effects on the relative equilibrium price to cancel each other out.

3.1.2 Contagion under DARA

Under more general terminal wealth and utility specifications, the dynamics of the n th relative price, $p_n(\omega, t)$, with respect to changes in the Brownian realization $\beta_k(\omega, t)$ when

$\sigma_{nk} = 0$ are surprisingly complex. For $k \neq m$, the partial derivative is given by

$$\frac{\partial p_n(\omega, t)}{\partial \beta_k(\omega, t)} = \left(\frac{e^{\mu_n T + \sigma_{nm} \beta_m(\omega, t)}}{P_0(\omega, t)^2 \sqrt{(T-t)(2\pi)^{2K}}} \right) \quad (17)$$

$$\text{Cov}_{y_k} \left[y_k, \mathbb{E}_{(\mathbf{x}, \mathbf{y}_{-k})} \left[\begin{array}{l} u'(W(\omega, t, \mathbf{y} + \sqrt{T-t} \sigma_{nm} \mathbf{e}_m)) u'(W(\omega, t, \mathbf{x})) \\ -u'(W(\omega, t, \mathbf{x} + \sqrt{T-t} \sigma_{nm} \mathbf{e}_m)) u'(W(\omega, t, \mathbf{y})) \end{array} \right] \right]$$

and its dynamics are rich enough to render, in fact, contagion in this representative agent economy a rather generic phenomenon. For, as I show below, under any DARA terminal-period utility, the relative price varies (monotonically) with the current realization of the k th Brownian motion even when the dispersion matrix Σ is diagonal and the terminal-period endowment is deterministic.

To actually stack the cards as much as possible against contagion, let us return to assuming that the two Brownian motions of interest operate through separate elements of the terminal wealth. Of course, this is a requirement that precludes either from being correlated with the terminal-period endowment. We restrict our attention, therefore, to the situation in which the k th and m th Brownian components affect only and different terminal dividends. Formally, the corresponding terminal wealth specification is given by

$$W(\omega, t, \mathbf{x}) = e(\omega, t, \mathbf{x}_{-(m,k)}) + \sum_{l \in N_k} D_l(\omega, t, \mathbf{x}_{-m}) + \sum_{i \in N_m} D_i(\omega, t, \mathbf{x}_{-k}) \quad (18)$$

As it turns out, under this specification, it is possible for the relative price of the n th risky security to depend monotonically on the current realization of the k th Brownian motion.

Proposition 3.1.3 *Suppose that the following conditions apply.*

- (i) *The agent's terminal utility exhibits DARA.*
- (ii) *The terminal dividend of the n th risky security is given by (12)*
- (iii) *For some $k \neq m$, the k th and m th Brownian motions affect only terminal dividends and different ones: $\frac{\partial e(\mathcal{I}(\omega, T))}{\partial \beta_k(\omega, T)} = 0 = \frac{\partial e(\mathcal{I}(\omega, T))}{\partial \beta_m(\omega, T)}$ and $N_k \cap N_m = \emptyset$.*
- (iv) *Same loadings for the m th Brownian motion: $\sigma_{n'm} = \sigma_{nm} \forall n' \in N_m$.*
- (v) *Agreement in the loadings of the k th Brownian motion: $\sigma_{n'k} \sigma_{n''k} > 0 \forall n', n'' \in N_k$.*

Then

$$\sigma_{n'k} \frac{\partial p_n(\omega, t)}{\partial \beta_k(\omega, t)} > 0 \quad \forall n' \in N_k$$

The three dispersion matrices below represent situations that fall within the operational realm of the proposition. In the first of these examples and for $k \geq 2$, the collection N_k of those securities whose terminal dividend is correlated with the k th Brownian motion is a singleton. Hence, the condition (iv) of the proposition is redundant in this case and the result applies for $k \geq 2$ as long as the terminal-period endowment is independent of either the first or the k th Brownian components. As, in addition, $s \frac{\partial p_n(\omega, t)}{\partial \beta_k(\omega, t)} > 0$ for $n = 1, 3$ (Theorem 2.1), we are able to sign the entire first and third rows of the dispersion matrix of relative prices.

In the second example, the result applies on the relative price of the first security and for $k \geq 2$ as long as the terminal-period endowment is independent of either the first or the k th Brownian components and $\sigma_{2k}\sigma_{3k} > 0$. This inequality is required by condition (iv) which is now binding as N_k is no longer a singleton. The same is true also in the last example where the relevant inequality becomes $\sigma_{2k}\sigma_{4k} > 0$. The proposition applies here for $k \geq 3$ and again as long as the terminal-endowment is independent of either the first or the k th Brownian motions. It cannot apply for $k = 2$ because condition (ii) is violated by the third asset, which belongs to $N_1 \cap N_2$. In either example, using also Theorem 2.1, we can sign the entire first row of the dispersion matrix of relative prices.

$$\Sigma_6 = \begin{pmatrix} s & 0 & 0 \\ 0 & \sigma_{22} & \sigma_{23} \\ s & 0 & 0 \end{pmatrix} \quad \sigma_{2k} \frac{\partial p_n(\omega, t)}{\partial \beta_k(\omega, t)} > 0 \quad n = 1, 3 \quad k = 2, 3$$

$$\Sigma_7 = \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & \sigma_{23} \\ 0 & \sigma_{32} & \sigma_{33} \end{pmatrix} \quad \sigma_{2k} \frac{\partial p_1(\omega, t)}{\partial \beta_k(\omega, t)} > 0 \quad k = 2, 3$$

$$\Sigma_8 = \begin{pmatrix} s & 0 & 0 & 0 \\ 0 & \sigma_{22} & \sigma_{23} & \sigma_{24} \\ s & \sigma_{32} & 0 & 0 \\ 0 & \sigma_{42} & \sigma_{43} & \sigma_{44} \end{pmatrix} \quad \sigma_{2k} \frac{\partial p_1(\omega, t)}{\partial \beta_k(\omega, t)} > 0 \quad k = 3, 4$$

An important subcase of the terminal wealth specification in (18) was that in (8), which restricted the k th and m th Brownian components to be correlated with only the terminal dividends the n' and n securities, respectively, for some $n' \in \{1, \dots, N\} \setminus \{n\}$. This requirement renders conditions (iii)-(iv) of Proposition 3.1.3 redundant and allows its claim to be stated as follows.

Corollary 3.1.4 *Let the following apply.*

(i) The agent's terminal utility exhibits DARA.

(ii) The terminal dividend of the n th risky security is given by (12)

(iii) For some $k \neq m$ and $n' \in \{1, \dots, N\} \setminus \{n\}$, the k th and m th Brownian motions affect, respectively, only the n' and n terminal dividends:

$$\frac{\partial e(\mathcal{I}(\omega, T))}{\partial \beta_k(\omega, T)} = 0 = \frac{\partial e(\mathcal{I}(\omega, T))}{\partial \beta_m(\omega, T)}, N_k = \{n'\}, \text{ and } N_m = \{n\}.$$

Then

$$\sigma_{n'k} \frac{\partial p_n(\omega, t)}{\partial \beta_k(\omega, t)} > 0$$

Notice that the conditions for this corollary to apply admit the case in which the dispersion matrix is diagonal, such as in the example below.

$$\Sigma_9 = \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix} \quad \sigma_{kk} \frac{\partial p_n(\omega, t)}{\partial \beta_k(\omega, t)} > 0 \quad n, k = 1, 2, 3$$

In this situation, the claim is valid for any security n and any Brownian motion $k \neq n$ as long as the terminal-period endowment is independent of both the n th and k th Brownian components. If, in particular, the terminal-period endowment is deterministic, then the corollary and Theorem 2.1 together allows us to sign the entire dispersion matrix of the relative price process. Specifically, with a diagonal dispersion matrix Σ and a deterministic terminal-period endowment, the corresponding terminal wealth specification is given by

$$W(\omega, t, \mathbf{x}) = e(T) + \sum_{i=1}^N D_i(\omega, t, x_i) \tag{19}$$

which is actually a special case of (15). In the corresponding dispersion matrix for the relative price process

$$J(\omega, t) = \left[\frac{\partial p_n(\omega, t)}{\partial \beta_k(\omega, t)} \right]_{(n,k) \in \{1, \dots, K\}^2}$$

the typical entry has the same sign as the respective factor loading σ_{nk} .

To conclude this section, I should point that Proposition 3.1.3 assumes the terminal wealth specification in (18) mainly for expositional ease in the presentation of its proof (see Appendix C). The result readily generalizes to the specification in (14) as long as each of the derivatives $\frac{\partial W(\omega, t, \mathbf{x}_{-m})}{\partial x_k}$ and $\frac{\partial W(\omega, t, x_m)}{\partial x_m}$ maintains a given sign on \mathbb{R} . Specifically, let $\lambda_k, \lambda_m \in \mathbb{R}$ be such that $\lambda_k \frac{\partial W(\omega, t, \mathbf{x}_{-m})}{\partial x_k}, \lambda_m \frac{\partial W(\omega, t, x_m)}{\partial x_m} > 0$ for all $\mathbf{x} \in \mathbb{R}^K$. As we already know, the wealth

effect of the realization $\beta_k(\omega, t)$ pushes both equilibrium prices $P_0(\omega, t)$ and $P_n(\omega, t)$ in the direction in which it moves the terminal wealth. Given the separability in (14), this direction is given by the derivative $\frac{\partial W(\beta_{-m}(\omega, T))}{\partial \beta_k(\omega, t)}$; i.e., by the sign of λ_k . By contrast, the specification in (14) being a special case of that in (??), the asset-riskiness effect of $\beta_k(\omega, t)$ on the relative price $p_n(\omega, t)$ is given - by (7) - as

$$\mathbb{E}_{x_k} \left[\text{Cov}_{\mathbf{x}_{-k}} [u''(W(\omega, t, \mathbf{x})), D_n(\omega, t, x_m)] \frac{\partial W(\omega, t, \mathbf{x}_{-m})}{\partial \beta_k(\omega, t)} \right]$$

As it turns out, the combined effect on the equilibrium relative price of the n th security is such that it changes monotonically. It is straightforward to reproduce the proof of Proposition 3.1.3 in this setting and verify that

$$\sigma_{nm} \lambda_m \lambda_k \frac{\partial p_n(\omega, t)}{\partial \beta_k(\omega, t)} > 0$$

3.2 Multi-dimensional Dividend Uncertainty

It remains to examine the general case in which the terminal dividend of the n th risky security varies with more than one Brownian motions. Let, therefore,

$$D_n(\mathcal{I}(\omega, T)) = e^{\mu_n T + \sigma_n^T (\beta(\omega, t) + \sqrt{T-t} \mathbf{x})} \quad (20)$$

and define also

$$K_n = \{m \in \{1, \dots, K\} : \sigma_{nm} \neq 0\}$$

to be the collection of those components of the Brownian vector that do affect the terminal dividend of the n th security. Unlike in the preceding section (where $K_n = \{m\}$), here this set will no longer be a singleton and, to facilitate the analysis, I will distinguish between whether or not K_n includes the k th Brownian component, with respect to whose current realization we are taking the derivative of the equilibrium relative price process.

3.2.1 Contagion

Suppose first that $k \notin K_n$. That is, $\sigma_{nk} = 0$ and the derivative of interest is given by

$$\frac{\partial p_n(\omega, t)}{\partial \beta_k(\omega, t)} = \left(\frac{e^{\mu_n T + \sigma_n^T \beta(\omega, t)}}{P_0(\omega, t)^2 \sqrt{(T-t)(2\pi)^{2K}}} \right) \quad (21)$$

$$\text{Cov}_{y_k} \left[y_k, \mathbb{E}_{(\mathbf{x}, \mathbf{y}_{-k})} \left[\begin{array}{c} u'(W(\omega, t, \mathbf{y} + \sqrt{T-t}\sigma_n)) u'(W(\omega, t, \mathbf{x})) \\ -u'(W(\omega, t, \mathbf{x} + \sqrt{T-t}\sigma_n)) u'(W(\omega, t, \mathbf{y})) \end{array} \right] \right]$$

As to be expected in the light of what we now know about K_n being a singleton, even though the expected terminal dividend of the n th risky security remains invariant with changes in the realization $\beta_k(\omega, t)$, its relative price may well be correlated with the k th Brownian motion. In what follows, I extend the settings underlying Propositions 3.1.2 and 3.1.3 and establish that this correlation is non-zero in the respective situations. More importantly perhaps, the sign of the correlation remains constant everywhere so that the relative price process $p_n(\omega, t)$ is monotone in $\beta_k(\omega, t)$.

Proposition 3.2.1 *Suppose that the following conditions apply.*

- (i) *The agent's terminal utility exhibits CARA.*
- (ii) *The terminal dividend of the n th risky security is given by (20)*
- (iii) *For some $k \notin K_n$, the k th Brownian motion affects only terminal dividends: $\frac{\partial e(\mathcal{I}(\omega, T))}{\partial \beta_k(\omega, T)} = 0$.*
- (iv) *Proportionality in factor loadings:*

$$\frac{\sigma_{n'm}}{\sigma_{nm}} = \lambda_{n'} \quad \forall (n', m) \in N_k \times K_n$$

- (v) *Agreement in the proportionality constants:*

$$\lambda_{n'} \sigma_{n'k} \lambda_{n''} \sigma_{n''k} > 0 \quad \forall n', n'' \in \cup_{m \in K_n} (N_m \cap N_k)$$

Then

$$\sigma_{n'm} \sigma_{nm} \sigma_{n'k} \frac{\partial p_n(\omega, t)}{\partial \beta_k(\omega, t)} < 0 \quad \forall n' \in N_m \cap N_k$$

To illustrate how this result applies, consider again the economy with the dispersion matrix Σ_3 for which we already know the signs of all the entries in the first row of the

dispersion matrix of relative prices. Proposition 3.2.1 allows us also to sign the element $\frac{\partial p_3(\omega, t)}{\partial \beta_2(\omega, t)}$. To this end, we observe first that $K_3 = \{1, 3\}$ and $N_2 = \{2\}$. Condition (iv) requires then the existence of some $\lambda_2 \in \mathbb{R}$ such that $\sigma_{31} = \lambda_2 \sigma_{21}$ and $\sigma_{33} = \lambda_2 \sigma_{23}$. Equivalently, that $\sigma_{31}/\sigma_{21} = \sigma_{33}/\sigma_{23}$. Condition (v), on the other hand, is redundant here; N_2 being a singleton, it coincides with the set $\cup_{m \in K_3} (N_m \cap N_2)$. Hence, as long as $\sigma_{31}/\sigma_{21} = \sigma_{33}/\sigma_{23}$ and the second Brownian motion is not correlated with the terminal period endowment, it must be $\sigma_{21} \sigma_{31} \sigma_{22} \frac{\partial p_3(\omega, t)}{\partial \beta_2(\omega, t)} < 0$. In contrast, the dispersion matrix Σ_8 corresponds to a situation in which the claim does not apply. Here, K_3 and N_2 are the same sets as before. Yet, condition (iv) cannot be satisfied.

The preceding claim was concerned with the case in which the agent exhibits CARA. Yet, even under more general utility specifications, we can construct settings in which the relative price process $p_n(\omega, t)$ is monotone in $\beta_k(\omega, t)$.

Proposition 3.2.2 *Suppose that the following conditions apply.*

- (i) *The agent's terminal utility exhibits DARA.*
- (ii) *The terminal dividend of the n th risky security is given by (20)*
- (iii) *For some $k \neq m$ and for all $m \in K_n$, the k th and m th Brownian motions affect only terminal dividends and different ones:*

$$\frac{\partial e(\mathcal{I}(\omega, T))}{\partial \beta_k(\omega, T)} = 0 = \frac{\partial e(\mathcal{I}(\omega, T))}{\partial \beta_m(\omega, T)}, \quad N_m \cap N_k = \emptyset \quad \forall m \in K_n$$

- (iv) *Same factor loadings for the dividends in N_m : $\sigma_{n'm} = \sigma_{nm} \forall (n', m) \in N_m \times K_n$.*
- (v) *Agreement in the loadings of the k th Brownian motion: $\sigma_{n'k} \sigma_{n''k} > 0 \forall n', n'' \in N_k$.*

Then

$$\sigma_{n'k} \frac{\partial p_n(\omega, t)}{\partial \beta_k(\omega, t)} > 0 \quad \forall n' \in N_k$$

Applying this result, we can sign the derivative of the relative price of the second security with respect to the first Brownian motion in the dispersion matrix Σ_6 . Of course, since $K_2 = \{2, 3\}$, condition (iii) above restricts us to the situation in which the terminal-period endowment is deterministic. On the other hand, condition (iv) is redundant in this example (for either $m \in K_2$, N_m consists of only the second security) whereas condition (v) is trivially satisfied ($N_1 = \{1, 3\}$ and $\sigma_{11} = \sigma_{31} = s$). In this case, it must be $s \frac{\partial p_2(\omega, t)}{\partial \beta_1(\omega, t)} > 0$. In the economy depicted by the matrix Σ_7 , the result can apply on the relative prices of the last

two securities with respect to the first Brownian motion. Again here, we need to assume that the terminal-period endowment is deterministic while condition (v) is redundant (N_1 is now a singleton). By contrast, condition (iv) requires that $\sigma_{22} = \sigma_{32}$ and $\sigma_{23} = \sigma_{33}$. Under these restrictions, $\sigma_{11} \frac{\partial p_n(\omega, t)}{\partial \beta_1(\omega, t)} > 0$ for $n = 2, 3$.

In the same fashion, we can identify the signs of some elements of the dispersion matrix of relative prices when the dispersion matrix is Σ_8 . As long as the terminal-period endowment is deterministic and, for $k = 2$, σ_{nk} is the same across all $n > 1$ while, for $k > 1$, this is the case for $n \in \{2, 4\}$, it must be $s \frac{\partial p_n(\omega, t)}{\partial \beta_1(\omega, t)} > 0$ for $n = 2, 4$ and $\sigma_{2k} \frac{\partial p_3(\omega, t)}{\partial \beta_k(\omega, t)} > 0$ for $k = 3, 4$.

3.2.2 General Dynamics

Turning to the case $k \in K_n$ ($\sigma_{nk} \neq 0$) exhausts the scope of my investigation on the dynamics of the relative price process. It is my hope to have, with my analysis thus far, made the point that, in general, the complexity of these dynamics is such that assertions about the direction of the relative price movements cannot be supported, except for particular situations, even when restricting attention to the case in which the terminal dividend of the security under study is not correlated with the source of uncertainty in question. In presenting this thesis, my strategy has been to find specifications, for the economic primitives of the particular asset market structure in this paper, under which the sign of the correlation between the relative price of the security and the underlying Brownian motion remains unambiguous throughout the stochastic domain. This achieves two things. By establishing that, as a norm, prices *are* correlated with underlying information even when payoffs are not, it indicates that the relative price dynamics are rich, much richer perhaps than one is led to expect at first glance armed with basic intuition. By showing, on the other hand, that it is by no means straightforward to identify settings in which the sign of this correlation remains constant, it attests to the complexity of the relative price dynamics.

Evidently, when the terminal dividend of the security is correlated with the Brownian dimension of interest, there is really little hope of pinpointing settings in which the correlation between the relative price of the security and the underlying Brownian motion maintains a constant sign throughout the stochastic domain. Nevertheless, I conclude this study by presenting a situation in which the relative price process is indeed monotone.

Proposition 3.2.3 *Let the following conditions apply.*

- (i) *The agent's terminal utility exhibits CRRA: $u(c) = \gamma c^\alpha$ $\alpha, \gamma < 0$ or $u(c) = \ln c$.*
- (ii) *The terminal dividend of the n th risky security is given by (20)*

(iii) $\exists \lambda_n \in \mathbb{R}_{++}$ s.t. $W(\mathbf{x} + \sqrt{T-t}\sigma_n) = \lambda_n W(\mathbf{x}) \forall \mathbf{x} \in \mathbb{R}^K$.

Then, setting $\alpha = 0$ in the logarithmic case,

$$\frac{\partial p_n(\omega, t)}{\partial \beta_k(\omega, t)} = \sigma_{nk} \lambda_n^{\alpha-1} e^{\mu_n T + \sigma_n^\top (\beta - \frac{(T-t)\sigma_n}{2})} \quad \forall k = 1, \dots, K$$

Even though the setting under which this result applies is admittedly quite specific, it allows the recovery of the entire dispersion matrix of relative prices; its n th row is given by

$$\mathbf{j}(\omega, t)_{p,n} = \lambda_n^{\alpha-1} e^{\mu_n T + \sigma_n^\top (\beta(\omega, t) - \frac{(T-t)\sigma_n}{2})} \sigma_n^\top$$

The workings of this proposition can be depicted by considering an economy in which (a) the terminal-period endowment is deterministic, $e(\mathcal{I}(\omega, T)) = 0$, and (b) the dispersion matrix Σ is such that $(\sigma_{n'} - \sigma_n)^\top \sigma_n = 0 \forall n' = 1, \dots, N$. It is straightforward to verify that these two restrictions suffice for condition (iii) of the proposition to apply (with $\lambda_n = e^{(T-t)\sigma_n^\top \sigma_n}$ in particular).¹⁹ Such an economy can be represented, for instance, by the dispersion matrix below.

$$\Sigma_{10} = \begin{pmatrix} \sigma_{11} & \cdot & \sigma_{1,m-1} & s_m & s_{m+1} & \sigma_{1,m+2} & \cdot & \sigma_{1K} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \sigma_{n-1,1} & \cdot & \sigma_{n-1,m-1} & s_m & s_{m+1} & \sigma_{n-1,m+2} & \cdot & \sigma_{n-1,K} \\ 0 & \dots & 0 & s_m & s_{m+1} & 0 & \dots & 0 \\ \sigma_{n+1,1} & \cdot & \sigma_{n+1,m-1} & s_m & s_{m+1} & \sigma_{n+1,m+2} & \cdot & \sigma_{n+1,K} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \sigma_{N,1} & \cdot & \sigma_{N,m-1} & s_m & s_{m+1} & \sigma_{N,m+2} & \cdot & \sigma_{NK} \end{pmatrix}$$

In this example, the m th and $m+1$ th Brownian motions affect all risky assets, with the same sensitivity, and the n th security is a “bet” on these two sources of macroeconomic uncertainty. Proposition 3.2.3 dictates here that $\sigma_{nk} \frac{\partial p_n(\omega, t)}{\partial \beta_k(\omega, t)} > 0$ for $k = m, m+1$.

Similarly, in the case where the relevant dispersion matrix is Σ_{11} , the claim would apply on the third security as long as $\sigma_{11}\sigma_{31} = \sigma_{22}\sigma_{32} = \sigma_{42}\sigma_{32} = \sigma_{31}^2 + \sigma_{32}^2$. That is, we would have $\sigma_{3k} \frac{\partial p_3(\omega, t)}{\partial \beta_k(\omega, t)} > 0$ for $k = 1, 2$. Observe, however, that, even though the dispersion matrix in this example is a generalization of Σ_8 , the claim cannot apply on the latter matrix. This is because condition (b) cannot be satisfied with $\sigma_{11} = \sigma_{31}$ unless $\sigma_{32} = 0$ (in which case,

¹⁹ $W(\mathbf{x} + \sqrt{T-t}\sigma_n) = \sum_{n'=1}^N e^{(T-t)\sigma_{n'}^\top \sigma_n} e^{\mu_{n'} T + \sigma_{n'}^\top (\beta + \sqrt{T-t}\mathbf{x})} = e^{(T-t)\sigma_n^\top \sigma_n} \sum_{n'=1}^N e^{\mu_{n'} T + \sigma_{n'}^\top (\beta + \sqrt{T-t}\mathbf{x})} = e^{(T-t)\sigma_n^\top \sigma_n} W(\mathbf{x})$.

of course, the terminal dividend of the third security would vary with only one Brownian component and the analysis of the previous section would be the relevant one).

$$\Sigma_{11} = \begin{pmatrix} \sigma_{11} & 0 & 0 & 0 \\ 0 & \sigma_{22} & \sigma_{23} & \sigma_{24} \\ \sigma_{31} & \sigma_{32} & 0 & 0 \\ 0 & \sigma_{42} & \sigma_{43} & \sigma_{44} \end{pmatrix} \quad \Sigma_{12} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$$

Another instructive example is given by the matrix Σ_{12} . Again here the derivative of the relative price of the first security, for instance, is such that $\sigma_{1k} \frac{\partial p_1(\omega, t)}{\partial \beta_k(\omega, t)} > 0$, with respect to either of the two Brownian motions, as long as the terminal-period endowment is deterministic and $\sigma_{11}(\sigma_{21} - \sigma_{11}) = \sigma_{12}(\sigma_{12} - \sigma_{22})$. Of course, if condition (b) holds also for the second security, the same result applies: $\sigma_{2k} \frac{\partial p_2(\omega, t)}{\partial \beta_k(\omega, t)} > 0$ for $k = 1, 2$. Yet, for this to happen, the dispersion matrix Σ must be degenerate. In general, actually, for (b) to be satisfied for all risky securities in the model, Σ ought to have identical rows, being of the form $\Sigma = (\sigma_1 \mathbf{e}, \dots, \sigma_K \mathbf{e})$ where $\mathbf{e} = \sum_{n=1}^N \mathbf{e}_n = (1, \dots, 1)^\top \in \mathbb{R}^N$.²⁰ In this case, even when markets are potentially dynamically complete ($N = K$), they will be necessarily dynamically incomplete. Proposition 3.2.3 would apply now to each and every risky security, restricting every row of the dispersion matrix of relative prices to be a multiple of the respective row of Σ . But then the dispersion matrix of relative prices would not be invertible for its determinant would be

$$|J_p(\omega, t)| = |\Sigma| \prod_{n=1}^K \lambda_n^{\alpha-1} e^{\mu_n T + \sigma_n^\top (\beta(\omega, t) - \frac{(T-t)\sigma_n}{2})}$$

and Σ is singular.²¹

4 Dynamic Completeness

In an Arrow-Debreu economy, the agents may shift consumption across states and time by trading a complete set of contingent claims. By contrast, in a securities market, they are constrained to trade a given set of securities. Such a market is said to be dynamically complete if the agents can still achieve any consumption allocation that would be feasible if

²⁰Selecting arbitrarily $n', n'' \in \{1, \dots, N\}$, condition (b) requires that $\sum_{k=1}^K \sigma_{n'k} (\sigma_{n''k} - \sigma_{n'k}) = 0 = \sum_{k=1}^K \sigma_{n''k} (\sigma_{n'k} - \sigma_{n''k})$. Equivalently, $\sum_{k=1}^K (\sigma_{n'k} - \sigma_{n''k})^2 = 0$ or $\sigma_{n'k} = \sigma_{n''k} \forall k = 1, \dots, K$.

²¹Of course, given Theorem 4.1, one can rule out here dynamic completeness immediately once one has observed that the underlying dispersion matrix Σ is singular.

there were instead a complete set of Arrow-Debreu contingent claims. Under continuous-time trading, when the information about the state of the world is revealed through a stochastic process, this may be possible by trading a given finite set of securities rapidly enough. In particular, when the underlying uncertainty is driven by Brownian motions, a necessary (yet by no means sufficient) condition for this to happen is that the securities market is potentially dynamically complete - i.e., the number of securities exceeds that of independent sources of uncertainty (independent Brownian motions) by at least one.²²

The existing literature on the existence and dynamic completeness of equilibrium in continuous-time finance models deals explicitly with the case in which markets are potentially dynamically complete. Apart from Anderson and Raimondo [4], every paper assumes that a given candidate equilibrium price process is dynamically complete and proceeds to establish that this candidate equilibrium is in fact an equilibrium.²³ However, their candidate equilibrium price processes are determined from the economic primitives of the model (the utility functions of the agents, their endowments, and the dividend processes of the securities) by a fixed point argument. And this means that, except in the extremely special cases where one can solve for the candidate equilibrium explicitly, it is not possible to verify from the primitives if the candidate equilibrium price process is indeed dynamically complete.

While the role of dynamic completeness is rather crucial in proving existence of equilibrium prices, it is actually fundamental when it comes to pricing financial derivatives. If the pricing process of the underlying securities is dynamically complete, then options and other derivative securities can be uniquely priced by arbitrage arguments and replicated by trading the underlying securities. In the absence of dynamic completeness, however, this is no longer the case; arbitrage considerations do not suffice to determine unique option prices and replication is not possible.

Given a financial environment, therefore, it is very important to be able to associate dynamic completeness with at least some of its economic fundamentals in a manner that remains unambiguously verifiable and holds generically across the space of these primitives. This is precisely the contribution of the following claim with respect to the securities market setting in this paper.

Theorem 4.1 *Let the securities market be potentially dynamically complete ($N = K$). The following are equivalent.*

²²When the underlying information process is not Brownian, the required number of securities may be larger.

²³Of course, the form of the assumption varies in the literature. See the introductory section of Anderson and Raimondo [4] for a summary review and discussion.

(i) The market is in fact dynamically complete.

(ii) Σ is nonsingular.

This result establishes a necessary and sufficient condition for dynamic completeness which depends only on the structure of the terminal dividends of the securities. None of the other primitives of the economic environment under study - in particular, neither the utility function nor the endowment of the representative agent - play a role. The condition is easily verified, by checking whether $|\Sigma| \neq 0$. It is also generically satisfied since, within the subset of \mathbb{R}^{2K} that forms the space of the matrix of factor loadings, the set of points where $|\Sigma| = 0$ is of zero-measure. This combination of generic validity and universal verifiability is quite rare in the literature. In most generic results on dynamic completeness, the condition for completeness is shown to hold except for a small set of the primitive parameters, being nevertheless difficult (if not impossible in some cases) to establish whether it does for particular values of these parameters.

Under the terminal dividend specification $D_n(\mathcal{I}(\omega, t)) = e^{\mu_n T + \sigma_n^T(\beta(\omega, t) + \sqrt{T-t}\mathbf{x})}$, its partial derivative with the realization at the node (ω, t) of the k th component of the Brownian vector is given by $\frac{\partial D_n(\mathcal{I}(\omega, t))}{\partial \beta_k(\mathcal{I}(\omega, t))} = \sigma_{nk} D_n(\mathcal{I}(\omega, t))$. That is, the general $N \times K$ Jacobian matrix of terminal dividends

$$J_D(\mathcal{I}(\omega, t)) = \begin{bmatrix} \frac{\partial D_1(\mathcal{I}(\omega, t))}{\partial \beta(\mathcal{I}(\omega, t))} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial D_N(\mathcal{I}(\omega, t))}{\partial \beta(\mathcal{I}(\omega, t))} \end{bmatrix} = \begin{bmatrix} \frac{\partial D_1(\mathcal{I}(\omega, t))}{\partial \beta_1(\mathcal{I}(\omega, t))} & \cdot & \cdot & \frac{\partial D_1(\mathcal{I}(\omega, t))}{\partial \beta_K(\mathcal{I}(\omega, t))} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \frac{\partial D_N(\mathcal{I}(\omega, t))}{\partial \beta_1(\mathcal{I}(\omega, t))} & \cdot & \cdot & \frac{\partial D_N(\mathcal{I}(\omega, t))}{\partial \beta_K(\mathcal{I}(\omega, t))} \end{bmatrix}$$

is constructed by multiplying each row of the matrix of factor loadings

$$\Sigma = [\sigma_{nk}]_{(n,k) \in \{1, \dots, N\} \times \{1, \dots, K\}}$$

by the corresponding terminal dividend. It follows, therefore, that the nondegeneracy condition (ii) of Theorem 4.1 is equivalent to requiring that the ($K \times K$ in this case) matrix $J_D(\mathcal{I}(\omega, t))$ be of full rank at every node (ω, t) . In other words, that the terminal dividends $D_1(\mathcal{I}(\omega, t)), \dots, D_K(\mathcal{I}(\omega, t))$ ought to be locally linearly independent at every node (ω, t) .²⁴

²⁴Recall that, if the matrix \tilde{A} results from multiplying a row of the square matrix A by the number λ , then $|\tilde{A}| = \lambda|A|$. In our case, $|J(\mathcal{I}(\omega, t))| = |\Sigma| \prod_{n=1}^K D_n(\mathcal{I}(\omega, t))$ with $\prod_{n=1}^K D_n(\mathcal{I}(\omega, t)) > 0$.

As is well-known, if there are enough securities for potential dynamic completeness, some form of linear independence amongst the securities' dividends is necessary for dynamic completeness of the Arrow-Debreu securities prices. In this sense, some form of linear independence amongst the dividends is (at least implicitly) assumed in any paper within the realm of the continuous-time finance literature that deals with the case of potentially dynamically complete markets. Of course, not all of these papers have lump terminal dividends and not all present the corresponding form of dividend linear independence explicitly. In fact, to the best of my knowledge, the only one that does both is Anderson and Raimondo [4], and their linear independence assumption is equivalent to the one I present here, for the corresponding setting.

Anderson and Raimondo [4] prove (in what is, in my view, the most complete to date treatment of existence for the setting they study) existence of equilibrium in a continuous-time securities market more general than the one I examine. They also study a single consumption good, pure exchange economy in which the uncertainty and the time structure for trade and consumption are exactly as here. Their typical security is in net supply $\eta_n \in \{0, 1\}$ and, in state ω , pays a dividend (measured in units of consumption) at a flow rate $g_n(\mathcal{I}(\omega, t))$ at times $t \in [0, T)$ and a lump amount $G_n(\mathcal{I}(\omega, T))$ at the terminal date. Their economy has many agents. The typical one gets endowed in state ω with the consumption good at a flow rate $f_i(\mathcal{I}(\omega, t))$ at times $t \in [0, T)$ and a lump amount $F_i(\mathcal{I}(\omega, T))$ at the end. Her preferences over consumption are given by the von Neumann-Morgenstern utility function

$$\mathbb{E}_\mu \left[\int_0^T h_i(c_i(\cdot, t), \mathcal{I}(\cdot, t)) dt + H_i(c_i(\cdot, T), \mathcal{I}(\cdot, T)) \right]$$

an expectation, over the state space Ω , of instantaneous utility functions of her measurable consumption process $c_i : [0, T] \times \Omega \mapsto \mathbb{R}_{++}$ and the process \mathcal{I} . The authors take the functions that apply on flows, $g_n, f_i : [0, T] \times \mathbb{R}^K \mapsto \mathbb{R}_+$ and $h_i : \mathbb{R}_+ \times ([0, T] \times \mathbb{R}^K) \mapsto \mathbb{R} \cup \{-\infty\}$, to be analytic on $(0, T) \times \mathbb{R}^K$ and $\mathbb{R}_+ \times ([0, T] \times \mathbb{R}^K)$, respectively.²⁵ The functions that apply on lump amounts, $G_n, F_i : \{T\} \times \mathbb{R}^K \mapsto \mathbb{R}_+$ and $H_i : \mathbb{R}_+ \times (\{T\} \times \mathbb{R}^K) \mapsto \mathbb{R} \cup \{-\infty\}$, are assumed to be, respectively, continuous almost everywhere on $\{T\} \times \mathbb{R}^K$ and C^2 on $\mathbb{R}_{++} \times (\{T\} \times \mathbb{R}^K)$. In addition, both utility functions h_i and H_i are required to satisfy certain standard regularity conditions.

Their securities market is potentially dynamically complete for they introduce $K + 1$

²⁵A function is said to be analytic if, at every point in its domain, there exists a power series which converges to the function on an open set containing the point.

securities, indexed by $n = 0, 1, \dots, K$. More importantly, they assume the following non-degeneracy condition on the terminal dividends of the securities: there exist (i) an open set $V \subset \mathbb{R}^K$ such that the terminal dividend of security 0 is positive if the terminal date realization of the Brownian vector falls within this set ($G_0(T, x) > 0 \forall \mathbf{x} \in V$) and (ii) some terminal date realization $\mathbf{x}^* \in V$ such that the Jacobian matrix

$$\begin{pmatrix} \frac{\partial}{\partial \beta} \left(\frac{G_1}{G_0} \right) |_{(T, \mathbf{x}^*)} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial}{\partial \beta} \left(\frac{G_K}{G_0} \right) |_{(T, \mathbf{x}^*)} \end{pmatrix}$$

has full rank. As it turns out, this exogenous assumption is sufficient for their equilibrium pricing process to be dynamically complete. As it happens, if the security 0 is a zero-coupon bond ($G_0(\omega, t) = 0$ for $t < T$ and $G_0(\omega, T) = 1$), their rank condition is nothing but the requirement that the $K \times K$ Jacobian matrix of terminal dividends

$$J_D(T, \mathbf{x}) = \begin{pmatrix} \frac{\partial G_1}{\partial \beta} |_{(T, \mathbf{x})} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial G_K}{\partial \beta} |_{(T, \mathbf{x})} \end{pmatrix}$$

is nonsingular at (T, \mathbf{x}^*) .

Clearly, the securities market that I analyze in this paper is a special case of that in Anderson and Raimondo [4]. For it obtains by restricting the functions g_n and f_i to be constant (at zero and one, respectively), hence, trivially analytic. Similarly, the terminal dividend and endowment specifications in this paper are both everywhere differentiable (hence, continuous) while all conventional state-independent utility functions satisfy the conditions Anderson and Raimondo impose on h_i and H_i .²⁶

Of course, their economy has many agents whereas I examine one with a representative agent. Yet, this does not matter with respect to a condition for dynamic completeness that is imposed on the structure of the securities dividends only. It is well-known that

²⁶The formulation of h_i and H_i allows for state-dependent instantaneous utility functions as long as the dependence enters through the process $\mathcal{I}(\omega, t)$. This is why the process enters as a potential separate functional argument in the text.

under dynamic completeness the financial equilibrium must be Pareto optimal which, in an economy with many agents, requires the existence of a (constant) vector of utility weights such that, at each node (ω, t) , the equilibrium consumptions maximize the weighted sum of the utilities of the agents. As this weighted sum is the utility of the representative agent and Theorem 4.1 applies regardless of its functional form (or that of the social endowment), its claim remains in force even with many agents.

In this sense, it is not surprising that Theorem 4.1 gives the same sufficient condition for dynamic completeness as Anderson and Raimondo do when one of their securities is a zero-coupon bond. Of course, in my formulation, when valid, the condition remains so universally rather than at some point of an open set. But this is because, under the terminal dividend specification in this paper, the nondegeneracy of the Jacobian matrix of the terminal dividends is globally equivalent to the nondegeneracy of Σ , a matrix of constants. Observe finally that, in the important special case of the Anderson and Raimondo setup that I examine, the “only if” direction of Theorem 4.1 complements their nondegeneracy hypothesis. It is also necessary for dynamic completeness.

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A Preliminary Results

Lemma A.1 Let $H : \mathbb{R} \times \mathbb{R}^K \mapsto \mathbb{R}$ be 2^{nd} -order differentiable and suppose that, for $\epsilon > 0$,

$$\sup_{\tilde{\beta} \in [\beta - \epsilon, \beta + \epsilon]} \sup_{\mathbf{x} \in \mathbb{R}^K} \left| \frac{\partial^2}{\partial \beta^2} H(\tilde{\beta}, \mathbf{x}) \right| \prod_{k=1}^K \max\{1, x_k^2\} \exp\left(-\frac{\mathbf{x}^\top \mathbf{x}}{2}\right) \equiv s(\beta) < +\infty$$

and $G : \mathbb{R} \mapsto \mathbb{R}$ given by

$$G(\tilde{\beta}) = \int_{\mathbb{R}^K} H(\tilde{\beta}, \mathbf{x}) d\Phi(\mathbf{x}) = \mathbb{E}_{\mathbf{x}} \left[H(\tilde{\beta}, \mathbf{x}) \right]$$

is defined for all $\tilde{\beta} \in [\beta - \epsilon, \beta + \epsilon]$. Then G is differentiable at β with

$$G'(\beta) = \int_{\mathbb{R}^K} \frac{\partial}{\partial \beta} H(\beta, \mathbf{x}) d\Phi(\mathbf{x}) = \mathbb{E}_{\mathbf{x}} \left[\frac{\partial}{\partial \beta} H(\beta, \mathbf{x}) \right]$$

Proof. For $h \in \mathbb{R} \setminus \{0\} : |h| < \epsilon$, we have

$$\begin{aligned} & \left| \frac{G(\beta + h) - G(\beta)}{h} - \int_{\mathbb{R}^K} \frac{\partial}{\partial \beta} H(\beta, \mathbf{x}) d\Phi(\mathbf{x}) \right| \\ &= \left| \int_{\mathbb{R}^K} \left(\frac{H(\beta + h, \mathbf{x}) - H(\beta, \mathbf{x})}{h} - \frac{\partial}{\partial \beta} H(\beta, \mathbf{x}) \right) d\Phi(\mathbf{x}) \right| \\ &\leq \int_{\mathbb{R}^K} \left| \frac{H(\beta + h, \mathbf{x}) - H(\beta, \mathbf{x})}{h} - \frac{\partial}{\partial \beta} H(\beta, \mathbf{x}) \right| d\Phi(\mathbf{x}) \\ &= \int_{\mathbb{R}^K} \left| \frac{\partial}{\partial \beta} H(\beta + \gamma(\mathbf{x})h, \mathbf{x}) - \frac{\partial}{\partial \beta} H(\beta, \mathbf{x}) \right| d\Phi(\mathbf{x}) \quad \text{for some } \gamma(\mathbf{x}) \in (0, 1) \\ &= \int_{\mathbb{R}^K} \left| h\gamma(\mathbf{x}) \frac{\partial^2}{\partial \beta^2} H(\beta + \delta(\mathbf{x})\gamma(\mathbf{x})h, \mathbf{x}) \right| d\Phi(\mathbf{x}) \quad \text{for some } \delta(\mathbf{x}) \in (0, 1) \\ &< |h| \int_{\mathbb{R}^K} \left| \frac{\partial^2}{\partial \beta^2} H(\beta + \delta(\mathbf{x})\gamma(\mathbf{x})h, \mathbf{x}) \right| d\Phi(\mathbf{x}) \leq \frac{|h| s(\beta)}{\sqrt{(2\pi)^K}} \int_{\mathbb{R}^K} \frac{d\mathbf{x}}{\prod_{k=1}^K \max\{1, x_k^2\}} \end{aligned}$$

where the second and third equalities above follow from the mean-value theorem, the first

inequality from the fact that $|\gamma(\mathbf{x})| < 1$ and the second by hypothesis. Since the x_k 's are independently distributed, however,

$$\int_{\mathbb{R}^K} \prod_{k=1}^K \frac{d\mathbf{x}}{\max\{1, x_k^2\}} = \prod_{k=1}^K \int_{\mathbb{R}} \frac{dx_k}{\max\{1, x_k^2\}} = \prod_{k=1}^K \left(\int_{-1}^1 dx_k + 2 \int_1^{+\infty} x_k^{-2} dx_k \right) = 4^K$$

and taking $|h| \rightarrow 0$ completes the proof. ■

The following is a well-known result.

Lemma A.2 *Let $\mathbf{z} \in \mathbb{R}^K$ be distributed $\mathcal{N}(\mathbf{0}, \mathbf{I}_K)$, $\theta \in \mathbb{R}^K$, and $g : \mathbb{R}^K \rightarrow \mathbb{R}$ s.t. $\mathbb{E}_{\mathbf{z}} [e^{\theta^\top \mathbf{z}} g(\mathbf{z})]$ exists. Then*

$$\mathbb{E}_{\mathbf{z}} [e^{\theta^\top \mathbf{z}} g(\mathbf{z})] = e^{\frac{\theta^\top \theta}{2}} \mathbb{E}_{\mathbf{z}} [g(\mathbf{z} + \theta)]$$

The next lemma makes use of the following result.

Claim A.0.1 *Let $\phi, \psi : \mathbb{R} \rightarrow \mathbb{R}$ be first-order differentiable functions such that the following integrals are defined*

- (i) $\int_{\mathbb{R}} \phi(z) \psi'(z) dz$ and $\int_{\mathbb{R}} \phi'(z) \psi(z) dz$
- (ii) $\int_{-\infty}^m \phi(z) \psi'(z) dz$ and $\int_{-\infty}^m \phi'(z) \psi(z) dz$, for some $m \in \mathbb{R}$
- (iii) $\int_l^{+\infty} \phi(z) \psi'(z) dz$, and $\int_l^{+\infty} \phi'(z) \psi(z) dz$, for some $l \in \mathbb{R}$.

Then

$$\int_{\mathbb{R}} \phi(z) \psi'(z) dz = \lim_{a \rightarrow +\infty} \phi(a) \psi(d) - \lim_{b \rightarrow -\infty} \phi(b) \psi(c) - \int_{\mathbb{R}} \phi'(z) \psi(z) dz$$

Proof. For the given $l, m \in \mathbb{R}$, we can write²⁷

$$\int_{\mathbb{R}} \phi(z) \psi'(z) dz = \int_{-\infty}^m \phi(z) \psi'(z) dz + \int_m^l \phi(z) \psi'(z) dz + \int_l^{+\infty} \phi(z) \psi'(z) dz$$

Using standard integration-by-parts, the proper integral above becomes

$$\int_m^l \phi(z) \psi'(z) dz = \phi(l) \psi(l) - \phi(m) \psi(m) - \int_m^l \phi'(z) \psi(z) dz$$

²⁷Since the integrals $\int_{\mathbb{R}} \phi(z) \psi'(z) dz$, $\int_{-\infty}^m \phi(z) \psi'(z) dz$, and $\int_l^{+\infty} \phi(z) \psi'(z) dz$ are all defined, so is the proper integral $\int_m^l \phi(z) \psi'(z) dz$.

The two improper integrals can be written as

$$\begin{aligned}
\int_{-\infty}^m \phi(z) \psi'(z) dz &= \lim_{b \rightarrow -\infty} \int_b^m \phi(z) \psi'(z) dz \\
&= \lim_{b \rightarrow -\infty} \left(\phi(m) \psi(m) - \phi(b) \psi(b) - \int_b^m \phi'(z) \psi(z) dz \right) \\
&= \phi(m) \psi(m) - \lim_{b \rightarrow -\infty} \phi(b) \psi(b) - \int_{-\infty}^m \phi'(z) \psi(z) dz
\end{aligned}$$

and

$$\begin{aligned}
\int_l^{+\infty} \phi(z) \psi'(z) dz &= \lim_{a \rightarrow +\infty} \int_l^a \phi(z) \psi'(z) dz \\
&= \lim_{a \rightarrow +\infty} \left(\phi(a) \psi(a) - \phi(l) \psi(l) - \int_l^a \phi'(z) \psi(z) dz \right) \\
&= \lim_{a \rightarrow +\infty} \phi(a) \psi(a) - \phi(l) \psi(l) - \int_l^{+\infty} \phi'(z) \psi(z) dz
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_{\mathbb{R}} \phi(z) \psi'(z) dz &= \lim_{a \rightarrow +\infty} \phi(a) \psi(a) - \lim_{b \rightarrow -\infty} \phi(b) \psi(b) \\
&\quad - \left(\int_{-\infty}^m \phi'(z) \psi(z) dz + \int_m^l \phi'(z) \psi(z) dz + \int_l^{+\infty} \phi'(z) \psi(z) dz \right) \\
&= \lim_{a \rightarrow +\infty} \phi(a) \psi(a) - \lim_{b \rightarrow -\infty} \phi(b) \psi(b) - \int_{\mathbb{R}} \phi'(z) \psi(z) dz
\end{aligned}$$

as required. ■

Lemma A.3 Let $\mathbf{z} \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_K)$, $\theta \in \mathbb{R}^K$, and $g : \mathbb{R}^K \mapsto \mathbb{R}$ s.t. the following conditions are met.

(i) $\mathbb{E}_{\mathbf{z}} \left[e^{\theta' \mathbf{z}} \frac{\partial g(\mathbf{z})}{\partial z_k} \right]$ and $\mathbb{E}_{\mathbf{z}} [z_k g(\mathbf{z} + \theta)]$ are defined.

(ii) For an arbitrary $\mathbf{z}_{-k} \in \mathbb{R}^{K-1}$, the preceding claim applies on the functions $\psi, \phi : \mathbb{R} \rightarrow \mathbb{R}$ given by $\psi(z_k) = g(z_k, \mathbf{z}_{-k})$ and $\phi(z_k) = e^{\theta' (z_k, \mathbf{z}_{-k}) - \frac{(z_k, \mathbf{z}_{-k})' (z_k, \mathbf{z}_{-k})}{2}}$.

(iii) $\lim_{z_k \rightarrow \pm\infty} \phi(z_k) \psi(z_k) = 0 \forall \mathbf{z}_{-k} \in \mathbb{R}^{K-1}$.

Then

$$\mathbb{E}_{\mathbf{z}} \left[e^{\theta' \mathbf{z}} \frac{\partial g(\mathbf{z})}{\partial z_k} \right] = e^{\frac{\theta' \theta}{2}} \mathbb{E}_{\mathbf{z}} [z_k g(\mathbf{z} + \theta)]$$

Proof. We have

$$\begin{aligned}\mathbb{E}_{\mathbf{z}} \left[e^{\theta^\top \mathbf{z}} \frac{\partial g(\mathbf{z})}{\partial z_k} \right] &= \int_{\mathbb{R}^K} e^{\theta^\top \mathbf{z}} \frac{\partial g(\mathbf{z})}{\partial z_k} d\Phi(\mathbf{z}) \\ &= \frac{1}{\sqrt{(2\pi)^K}} \int_{\mathbb{R}^{K-1}} \left(\int_{\mathbb{R}} e^{\theta^\top \mathbf{z}} \frac{\partial g(\mathbf{z})}{\partial z_k} e^{-\frac{z_k^2}{2}} dz_k \right) e^{-\frac{\sum_{i \neq k} z_i^2}{2}} d\mathbf{z}_{-k}\end{aligned}$$

By the preceding claim, we can use integration by parts to simplify the integral in the brackets. Specifically, given $\mathbf{z}_{-k} \in \mathbb{R}^{K-1}$ and the functions $\phi : \mathbb{R} \rightarrow \mathbb{R}_+$ and $\psi : \mathbb{R} \rightarrow \mathbb{R}$ as in the proposition, we have

$$\begin{aligned}& \int_{\mathbb{R}} e^{\theta^\top \mathbf{z}} \frac{\partial g(\mathbf{z})}{\partial z_k} e^{-\frac{z_k^2}{2}} dz_k \\ &= \int_{\mathbb{R}} \phi(z_k) \psi'(z_k) dz_k \\ &= \lim_{a \rightarrow +\infty} \phi(a) \psi(a) - \lim_{b \rightarrow -\infty} \phi(b) \psi(b) - \int_{\mathbb{R}} \phi'(z) \psi(z) dz \\ &= \left(\lim_{z_k \rightarrow +\infty} e^{\theta^\top \mathbf{z} - \frac{z_k^2}{2}} g(\mathbf{z}) - \lim_{z_k \rightarrow -\infty} e^{\theta^\top \mathbf{z} - \frac{z_k^2}{2}} g(\mathbf{z}) \right) - \int_{\mathbb{R}} (\theta_k - z_k) g(\mathbf{z}) e^{\theta^\top \mathbf{z} - \frac{z_k^2}{2}} dz_k \\ &= \int_{\mathbb{R}} (z_k - \theta_k) g(\mathbf{z}) e^{\theta^\top \mathbf{z} - \frac{z_k^2}{2}} dz_k\end{aligned}$$

Integrating now over $\mathbf{z}_{-k} \in \mathbb{R}^{K-1}$ gives

$$\begin{aligned}& \int_{\mathbb{R}^{K-1}} \left(\int_{\mathbb{R}} e^{\theta^\top \mathbf{z}} \frac{\partial g(\mathbf{z})}{\partial z_k} e^{-\frac{z_k^2}{2}} dz_k \right) e^{-\frac{\sum_{i \neq k} z_i^2}{2}} d\mathbf{z}_{-k} \\ &= \int_{\mathbb{R}^{K-1}} \left(\int_{\mathbb{R}} (z_k - \theta_k) g(\mathbf{z}) e^{\theta^\top \mathbf{z} - \frac{z_k^2}{2}} dz_k \right) e^{-\frac{\sum_{i \neq k} z_i^2}{2}} d\mathbf{z}_{-k} \\ &= e^{\frac{\theta^\top \theta}{2}} \int_{\mathbb{R}^K} (z_k - \theta_k) g(\mathbf{z}) e^{-\frac{\sum_i (z_i - \theta_i)^2}{2}} d\mathbf{z} \\ &= e^{\frac{\theta^\top \theta}{2}} \int_{\mathbb{R}^K} (z_k - \theta_k) g(\mathbf{z}) e^{-\frac{(\mathbf{z} - \theta)^\top (\mathbf{z} - \theta)}{2}} d\mathbf{z} = e^{\frac{\theta^\top \theta}{2}} \int_{\mathbb{R}^K} z_k g(\mathbf{z} + \theta) e^{-\frac{\mathbf{z}^\top \mathbf{z}}{2}} d\mathbf{z}\end{aligned}$$

The required result follows immediately. ■

Lemma A.4 For $n \in \mathbb{N} \setminus \{0\}$, let $\mathbf{S} \subseteq \mathbb{R}^n$ be of non-zero Lebesgue measure and such that $\mathbf{S} \times \mathbf{S}$ is symmetric around the origin.²⁸ Suppose that the following conditions are satisfied.

(i) $g : \mathbf{S} \times \mathbf{S} \mapsto \mathbb{R}_+$ is symmetric - i.e., $g(\mathbf{x}, \mathbf{y}) = g(\mathbf{y}, \mathbf{x})$ - everywhere on its domain

²⁸ $\mathbf{S} \times \mathbf{S}$ symmetric around the origin $\mathbf{0} \in \mathbb{R}^{2n}$ means that $\langle (\mathbf{x}, \mathbf{y}) \in \mathbf{S} \times \mathbf{S} \implies (\mathbf{y}, \mathbf{x}) \in \mathbf{S} \times \mathbf{S} \rangle$ holds $\forall (\mathbf{x}, \mathbf{y}) \in \mathbf{S} \times \mathbf{S}$.

except for sets of measure zero.²⁹

(ii) $f : \mathbf{S} \times \mathbf{S} \mapsto \mathbb{R}$ is such that $f(\mathbf{x}, \mathbf{y}) + f(\mathbf{y}, \mathbf{x}) \geq 0$ everywhere on its domain except for sets of measure zero.

(iii) $g(\cdot) f(\cdot)$ is Lebesgue-integrable over $\mathbf{S} \times \mathbf{S}$.

Then

$$\int_{\mathbf{S} \times \mathbf{S}} g(\mathbf{x}, \mathbf{y}) f(\mathbf{x}, \mathbf{y}) d(\mathbf{x}, \mathbf{y}) \geq 0$$

with strict inequality iff $g(\mathbf{x}, \mathbf{y}) [f(\mathbf{x}, \mathbf{y}) + f(\mathbf{y}, \mathbf{x})] \neq 0$ on a subset of $\mathbf{S} \times \mathbf{S}$ of non-zero measure.

Proof. Since $(gf)(\cdot)$ is integrable, by the Fubini-Tonelli theorem, the integral in question can be written as an iterated one:

$$\int_{\mathbf{S} \times \mathbf{S}} g(\mathbf{x}, \mathbf{y}) f(\mathbf{x}, \mathbf{y}) d(\mathbf{x}, \mathbf{y}) = \int_{\mathbf{S}} \left(\int_{\mathbf{S}} g(\mathbf{x}, \mathbf{y}) f(\mathbf{x}, \mathbf{y}) d\mathbf{y} \right) d\mathbf{x}$$

and, by re-naming the variables of integration, we can write it also as

$$\begin{aligned} \int_{\mathbf{S} \times \mathbf{S}} g(\mathbf{x}, \mathbf{y}) f(\mathbf{x}, \mathbf{y}) d(\mathbf{x}, \mathbf{y}) &= \int_{\mathbf{S} \times \mathbf{S}} g(\mathbf{y}, \mathbf{x}) f(\mathbf{y}, \mathbf{x}) d(\mathbf{y}, \mathbf{x}) \\ &= \int_{\mathbf{S}} \left(\int_{\mathbf{S}} g(\mathbf{y}, \mathbf{x}) f(\mathbf{y}, \mathbf{x}) d\mathbf{y} \right) d\mathbf{x} \end{aligned}$$

Hence,

$$\begin{aligned} &2 \int_{\mathbf{S} \times \mathbf{S}} g(\mathbf{x}, \mathbf{y}) f(\mathbf{x}, \mathbf{y}) d(\mathbf{x}, \mathbf{y}) \\ &= \int_{\mathbf{S}} \left(\int_{\mathbf{S}} g(\mathbf{x}, \mathbf{y}) f(\mathbf{x}, \mathbf{y}) d\mathbf{y} \right) d\mathbf{x} + \int_{\mathbf{S}} \left(\int_{\mathbf{S}} g(\mathbf{y}, \mathbf{x}) f(\mathbf{y}, \mathbf{x}) d\mathbf{y} \right) d\mathbf{x} \\ &= \int_{\mathbf{S}} \left(\int_{\mathbf{S}} g(\mathbf{x}, \mathbf{y}) [f(\mathbf{x}, \mathbf{y}) + f(\mathbf{y}, \mathbf{x})] d\mathbf{y} \right) d\mathbf{x} \geq 0 \end{aligned}$$

Obviously, the inequality is strict if and only if $g(\mathbf{x}, \mathbf{y}) [f(\mathbf{x}, \mathbf{y}) + f(\mathbf{y}, \mathbf{x})] \neq 0$ on a subset of $\mathbf{S} \times \mathbf{S}$ of positive measure. ■

²⁹More generally, the lemma holds for any $g : \mathbf{S} \times \mathbf{S} \mapsto \mathbb{R}$ that is non-negative and symmetric almost everywhere.

Lemma A.5 *Let a random vector $\mathbf{x} \in \mathbb{R}^K$ and a function $h : \mathbb{R}^K \mapsto \mathbb{R}$ be such that $\mathbb{E}_{\mathbf{x}} [h(\mathbf{x})]$ and $\mathbb{E}_{\mathbf{x}} [x_k h(\mathbf{x})]$ are well-defined with $\mathbb{E}_{\mathbf{x}} [h(\mathbf{x})] \neq 0$. Let also $f : \mathbb{R} \mapsto \mathbb{R}$ be given by $f(y_k) = \mathbb{E}_{\mathbf{x}} [(y_k - x_k) h(\mathbf{x})]$. Then,*

$$\exists y_k^0 \in \mathbb{R} : (y_k - y_k^0) f(y_k) \mathbb{E}_{\mathbf{x}} [h(\mathbf{x})] > 0 \quad \forall y_k \in \mathbb{R} \setminus \{y_k^0\}$$

Proof. Given that $\mathbb{E}_{\mathbf{x}} [h(\mathbf{x})] \neq 0$, we can write

$$f(y_k) = \mathbb{E}_{\mathbf{x}} [h(\mathbf{x})] \left(y_k - \frac{\mathbb{E}_{\mathbf{x}} [x_k h(\mathbf{x})]}{\mathbb{E}_{\mathbf{x}} [h(\mathbf{x})]} \right)$$

and it suffices to define $y_k^0 = \mathbb{E}_{\mathbf{x}} [x_k h(\mathbf{x})] / \mathbb{E}_{\mathbf{x}} [h(\mathbf{x})]$. ■

B Comonotonicity and Covariance

For a set \mathbf{S} and an algebra σ on \mathbf{S} , let (\mathbf{S}, σ) be a probabilizable space and $B(\mathbf{S}, \mathbb{R})$ be the set of bounded σ -measurable functions $\mathbf{S} \mapsto \mathbb{R}$. Two random variables $g, f \in B(\mathbf{S}, \mathbb{R})$ are said to be comonotonic if

$$[g(\mathbf{x}) - g(\mathbf{y})][f(\mathbf{x}) - f(\mathbf{y})] \geq 0 \quad \forall \mathbf{x}, \mathbf{y} \in \mathbf{S}$$

They are strictly comonotonic if the inequality is strict whenever $\mathbf{x} \neq \mathbf{y}$. The following result is borrowed from the appendix of [?]. I present the relevant for my argument “only if” part of the proof.

Lemma B.1 *The following are equivalent.*

- (i) $g, f \in B(\mathbf{S}, \mathbb{R})$ are (strictly) comonotonic
- (ii) $\text{Cov}_{\mu} [g, f] \geq 0$ (> 0) for any probability measure μ on (\mathbf{S}, σ) .

Proof. Let g and f be comonotonic and μ a probability measure on (\mathbf{S}, σ) . We have

$$\begin{aligned}
2\text{Cov}_\mu [g, f] &= 2 (\mathbb{E}_\mu [gf] - \mathbb{E}_\mu [g] \mathbb{E}_\mu [f]) \\
&= 2 \left(\int_{\mathbf{S}} g(\mathbf{x}) f(\mathbf{x}) d\mu(\mathbf{x}) - \int_{\mathbf{S}} g(\mathbf{y}) d\mu(\mathbf{y}) \int_{\mathbf{S}} f(\mathbf{x}) d\mu(\mathbf{x}) \right) \\
&= \int_{\mathbf{S}} g(\mathbf{x}) f(\mathbf{x}) d\mu(\mathbf{x}) + \int_{\mathbf{S}} g(\mathbf{y}) f(\mathbf{y}) d\mu(\mathbf{y}) \\
&\quad - \int_{\mathbf{S}} g(\mathbf{y}) d\mu(\mathbf{y}) \int_{\mathbf{S}} f(\mathbf{x}) d\mu(\mathbf{x}) - \int_{\mathbf{S}} g(\mathbf{x}) d\mu(\mathbf{x}) \int_{\mathbf{S}} f(\mathbf{y}) d\mu(\mathbf{y}) \\
&= \int_{\mathbf{S} \times \mathbf{S}} [g(\mathbf{x}) - g(\mathbf{y})] [f(\mathbf{x}) - f(\mathbf{y})] d\mu(\mathbf{x}) d\mu(\mathbf{y}) \\
&\geq 0
\end{aligned}$$

where the third equality uses a change of the variables of integration. The validity of the claim when the comonotonicity is strict is obvious. ■

Regarding the application of this result in the main text, two points should be made clear. First, f and g need not be bounded. The boundedness condition guarantees that the integrals above exist for *any* probability measure μ on (S, σ) . In the analysis of the asset-riskness effect, I fix $\mathbf{y} \in \mathbb{R}^{K-M}$ taking $\mathbf{z} \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_M)$, $f : \mathbb{R}^M \mapsto \mathbb{R}_{++}$ and $g : \mathbb{R}^M \mapsto \mathbb{R}_{--}$ with $f(\mathbf{z}) = e^{\mu_n T + \sigma_n^1 (\beta(\omega, t) + \sqrt{T-t} \mathbf{z})}$ and $g(\mathbf{z}) = u''(F(\beta(\omega, t), (\mathbf{z}, \mathbf{y})))$. The relevant expectations are well-defined even though f and g are, respectively, not and not necessarily bounded. Second, non-increasing absolute risk aversion, $r'_A(\cdot) \leq 0$, implies that $u'''(\cdot) > 0$. This guarantees in turn that, other things being equal, $f(\cdot)$ and $g(\cdot)$ are strictly comonotonic. For the terminal-period wealth process $F(\beta(\omega, t), (\mathbf{z}, \mathbf{y}))$ in (??) is strictly increasing in the realization of the terminal-period dividend, $e^{\mu_j T + \sigma_j^1 (\beta(\omega, t) + \sqrt{T-t} \mathbf{z})}$.

C Proofs of the Results in the Text

This section presents the proofs for the various results in the paper. With respect to notation, keep in mind that, purely for reasons of expositional ease, my expressions will no longer display the current node (ω, t) of the Brownian tree. That is, I will be writing β , β_k , and β_m for $\beta(\omega, t)$, $\beta_k(\omega, t)$, and $\beta_m(\omega, t)$, respectively. Also, P_0 , P_n , and p_n instead of $P_0(\omega, t)$, $P_n(\omega, t)$, and $p_n(\omega, t)$.

Theorem 2.1

For $n \in \{1, \dots, N\}$ and $k \in \{1, \dots, K\}$, consider (4). The terms $\frac{\partial p_n}{\partial \beta_k}$ and $\frac{\partial P_0}{\partial \beta_k}$ apply the partial-derivative operator $\frac{\partial}{\partial \beta_k}$ on $P_n = \mathbb{E}_{\mathbf{x}} \left[u' (F(\mathbf{x})) e^{\mu_n T + \sigma_n^{\top} (\beta + \sqrt{T-t} \mathbf{x})} \right]$ and $P_0 = \mathbb{E}_{\mathbf{x}} [u' (F(\mathbf{x}))]$, respectively. Lemma A.1 guarantees that the partial-derivative operator commutes with the expectations operator in this case. Hence, the partial derivative terms on the right-hand side of (4) may be written as follows

$$\begin{aligned} \frac{\partial P_n}{\partial \beta_k} &= \mathbb{E}_{\mathbf{x}} \left[\frac{\partial}{\partial \beta_k} \left(u' (F(\mathbf{x})) e^{\mu_n T + \sigma_n^{\top} (\beta + \sqrt{T-t} \mathbf{x})} \right) \right] \\ &= \sigma_{nk} \mathbb{E}_{\mathbf{x}} \left[u' (F(\mathbf{x})) e^{\mu_n T + \sigma_n^{\top} (\beta + \sqrt{T-t} \mathbf{x})} \right] \\ &\quad + \mathbb{E}_{\mathbf{x}} \left[u'' (F(\mathbf{x})) e^{\mu_n T + \sigma_n^{\top} (\beta + \sqrt{T-t} \mathbf{x})} \frac{\partial F(\mathbf{x})}{\partial \beta_k} \right] \\ \frac{\partial P_0}{\partial \beta_k} &= \mathbb{E}_{\mathbf{x}} \left[u'' (F(\mathbf{x})) \frac{\partial F(\mathbf{x})}{\partial \beta_k} \right] \end{aligned}$$

Using Lemma A.2, moreover, we get

$$\begin{aligned} p_n \equiv \frac{P_n}{P_0} &= e^{\mu_n T + \sigma_n^{\top} (\beta + \frac{(T-t)}{2} \sigma_n)} \frac{\mathbb{E}_{\mathbf{x}} [u' (F(\mathbf{x} + \sqrt{T-t} \sigma_n))] }{\mathbb{E}_{\mathbf{x}} [u' (F(\mathbf{x}))]} \\ \frac{\partial P_n}{\partial \beta_k} &= e^{\mu_n T + \sigma_n^{\top} (\beta + \frac{(T-t)}{2} \sigma_n)} \left(\begin{array}{c} \sigma_{nk} \mathbb{E}_{\mathbf{x}} [u' (F(\mathbf{x} + \sqrt{T-t} \sigma_n))] \\ + \mathbb{E}_{\mathbf{x}} \left[u'' (F(\mathbf{x} + \sqrt{T-t} \sigma_n)) \frac{\partial F(\mathbf{x} + \sqrt{T-t} \sigma_n)}{\partial \beta_k} \right] \end{array} \right) \end{aligned}$$

Hence,

$$\begin{aligned} \frac{P_0^2}{e^{\mu_n T + \sigma_n^{\top} \beta}} \frac{\partial p_n}{\partial \beta_k} &= \sigma_{nk} \mathbb{E}_{\mathbf{x}} \left[u' (F(\mathbf{x})) e^{\sqrt{T-t} \sigma_n^{\top} \mathbf{x}} \right] \mathbb{E}_{\mathbf{x}} [u' (F(\mathbf{x}))] \\ &\quad + \mathbb{E}_{\mathbf{x}} [u' (F(\mathbf{x}))] \mathbb{E}_{\mathbf{x}} \left[u'' (F(\mathbf{x})) e^{\sqrt{T-t} \sigma_n^{\top} \mathbf{x}} \frac{\partial F(\mathbf{x})}{\partial \beta_k} \right] \\ &\quad - \mathbb{E}_{\mathbf{x}} \left[u' (F(\mathbf{x})) e^{\sqrt{T-t} \sigma_n^{\top} \mathbf{x}} \right] \mathbb{E}_{\mathbf{x}} \left[u'' (F(\mathbf{x})) \frac{\partial F(\mathbf{x})}{\partial \beta_k} \right] \\ &= \sigma_{nk} \mathbb{E}_{\mathbf{y}} \left[u' (F(\mathbf{y})) e^{\sqrt{T-t} \sigma_n^{\top} \mathbf{y}} \right] \mathbb{E}_{\mathbf{x}} [u' (F(\mathbf{x}))] \\ &\quad + \mathbb{E}_{\mathbf{x}} [u' (F(\mathbf{x}))] \mathbb{E}_{\mathbf{y}} \left[u'' (F(\mathbf{y})) e^{\sqrt{T-t} \sigma_n^{\top} \mathbf{y}} \frac{\partial F(\mathbf{y})}{\partial \beta_k} \right] \\ &\quad - \mathbb{E}_{\mathbf{y}} \left[u' (F(\mathbf{y})) e^{\sqrt{T-t} \sigma_n^{\top} \mathbf{y}} \right] \mathbb{E}_{\mathbf{x}} \left[u'' (F(\mathbf{x})) \frac{\partial F(\mathbf{x})}{\partial \beta_k} \right] \end{aligned} \tag{22}$$

the second equality using a re-naming of variables of integration with $\mathbf{y}, \mathbf{x} \sim \text{i.i.d. } \mathcal{N}(\mathbf{0}, \mathbf{I}_K)$. For the terminal-period wealth, on the other hand, we have

$$\begin{aligned}
\frac{\partial F(\mathbf{x})}{\partial \beta_k} &= \frac{\partial}{\partial \beta_k} \left(\rho(\beta + \sqrt{T-t}\mathbf{x}) + \sum_{i=1}^N e^{\mu_i T + \sigma_i^\top (\beta + \sqrt{T-t}\mathbf{x})} \right) \\
&= \frac{\partial \rho(\beta + \sqrt{T-t}\mathbf{x})}{\partial \beta_k} + \sum_{i=1}^N \sigma_{ik} e^{\mu_i T + \sigma_i^\top (\beta + \sqrt{T-t}\mathbf{x})} \\
&= \frac{1}{\sqrt{T-t}} \frac{\partial}{\partial x_k} \left(\rho(\beta + \sqrt{T-t}\mathbf{x}) + \sum_{i=1}^N e^{\mu_i T + \sigma_i^\top (\beta(\omega, t) + \sqrt{T-t}\mathbf{x})} \right) \\
&= \frac{1}{\sqrt{T-t}} \frac{\partial F(\mathbf{x})}{\partial x_k}
\end{aligned} \tag{23}$$

(22), therefore, can be re-written as

$$\begin{aligned}
\frac{P_0^2}{e^{\mu_n T + \sigma_n^\top \beta_n}} \frac{\partial p_n}{\partial \beta_k} &= \sigma_{nk} \mathbb{E}_{\mathbf{y}} \left[u'(F(\mathbf{y})) e^{\sqrt{T-t}\sigma_n^\top \mathbf{y}} \right] \mathbb{E}_{\mathbf{x}} [u'(F(\mathbf{x}))] \\
&\quad + \frac{1}{\sqrt{T-t}} \mathbb{E}_{\mathbf{y}} \left[e^{\sqrt{T-t}\sigma_n^\top \mathbf{y}} \frac{\partial u'(F(\mathbf{y}))}{\partial y_k} \right] \mathbb{E}_{\mathbf{x}} [u'(F(\mathbf{x}))] \\
&\quad - \frac{1}{\sqrt{T-t}} \mathbb{E}_{\mathbf{x}} \left[\frac{\partial u'(F(\mathbf{x}))}{\partial x_k} \right] \mathbb{E}_{\mathbf{y}} \left[u'(F(\mathbf{y})) e^{\sqrt{T-t}\sigma_n^\top \mathbf{y}} \right]
\end{aligned}$$

Apply now Lemma A.2 on the term $\mathbb{E}_{\mathbf{y}} \left[u'(F(\mathbf{y})) e^{\sqrt{T-t}\sigma_n^\top \mathbf{y}} \right]$ and Lemma A.3 on each of $\mathbb{E}_{\mathbf{y}} \left[e^{\sqrt{T-t}\sigma_n^\top \mathbf{y}} \frac{\partial u'(F(\mathbf{y}))}{\partial y_k} \right]$ and $\mathbb{E}_{\mathbf{x}} \left[\frac{\partial u'(F(\mathbf{x}))}{\partial x_k} \right]$ (setting, for the latter term, $\theta = \mathbf{0}$ in Lemma A.3). The last equation gives

$$\begin{aligned}
\frac{\sqrt{T-t} P_0^2}{e^{\mu_n T + \sigma_n^\top (\beta - \frac{(T-t)\sigma_n}{2})}} \frac{\partial p_n}{\partial \beta_k} &= \sqrt{T-t} \sigma_{nk} \mathbb{E}_{\mathbf{y}} \left[u' \left(F \left(\mathbf{y} + \sqrt{T-t}\sigma_n \right) \right) \right] \mathbb{E}_{\mathbf{x}} [u'(F(\mathbf{x}))] \\
&\quad + \mathbb{E}_{\mathbf{y}} \left[y_k u' \left(F \left(\mathbf{y} + \sqrt{T-t}\sigma_n \right) \right) \right] \mathbb{E}_{\mathbf{x}} [u'(F(\mathbf{x}))] \\
&\quad - \mathbb{E}_{\mathbf{x}} [x_k u'(F(\mathbf{x}))] \mathbb{E}_{\mathbf{y}} \left[u' \left(F \left(\mathbf{y} + \sqrt{T-t}\sigma_n \right) \right) \right] \\
&= \mathbb{E}_{\mathbf{y}} \left[\left(y_k + \sqrt{T-t}\sigma_{nk} \right) u' \left(F \left(\mathbf{y} + \sqrt{T-t}\sigma_n \right) \right) \right] \mathbb{E}_{\mathbf{x}} [u'(F(\mathbf{x}))] \\
&\quad - \mathbb{E}_{\mathbf{x}} [x_k u'(F(\mathbf{x}))] \mathbb{E}_{\mathbf{y}} \left[u' \left(F \left(\mathbf{y} + \sqrt{T-t}\sigma_n \right) \right) \right] \\
&= \mathbb{E}_{\tilde{\mathbf{y}}} [\tilde{y}_k u'(F(\tilde{\mathbf{y}}))] \mathbb{E}_{\mathbf{x}} [u'(F(\mathbf{x}))] - \mathbb{E}_{\mathbf{x}} [x_k u'(F(\mathbf{x}))] \mathbb{E}_{\tilde{\mathbf{y}}} [u'(F(\tilde{\mathbf{y}}))]
\end{aligned} \tag{24}$$

where $\tilde{\mathbf{y}} \sim \mathcal{N}(\sqrt{T-t}\sigma_n, \mathbf{I}_K)$ is independent of \mathbf{x} . That is,

$$\begin{aligned} & \frac{P_0^2 \sqrt{T-t} (2\pi)^K}{e^{\mu_n T + \sigma_n^\top (\beta + \frac{T-t}{2}\sigma_n)}} \frac{\partial p_n}{\partial \beta_k} \\ &= \int_{\mathbb{R}^{2K}} u'(F(\tilde{\mathbf{y}})) u'(F(\mathbf{x})) \tilde{y}_k e^{-\frac{(\tilde{\mathbf{y}} - \sqrt{T-t}\sigma_n)^\top (\tilde{\mathbf{y}} - \sqrt{T-t}\sigma_n) + \mathbf{x}^\top \mathbf{x}}{2}} d\mathbf{x} d\tilde{\mathbf{y}} \\ & \quad - \int_{\mathbb{R}^{2K}} u'(F(\tilde{\mathbf{y}})) u'(F(\mathbf{x})) x_k e^{-\frac{(\tilde{\mathbf{y}} - \sqrt{T-t}\sigma_n)^\top (\tilde{\mathbf{y}} - \sqrt{T-t}\sigma_n) + \mathbf{x}^\top \mathbf{x}}{2}} d\mathbf{x} d\tilde{\mathbf{y}} \end{aligned}$$

Equivalently, changing variables of integration,

$$\frac{P_0^2 \sqrt{T-t} (2\pi)^K}{e^{\mu_n T + \sigma_n^\top \beta}} \frac{\partial p_n}{\partial \beta_k} = \int_{\mathbb{R}^{2K}} u'(F(\mathbf{y})) u'(F(\mathbf{x})) (y_k - x_k) e^{\sqrt{T-t}\sigma_n^\top \mathbf{y} - \frac{\mathbf{y}^\top \mathbf{y} + \mathbf{x}^\top \mathbf{x}}{2}} d\mathbf{x} d\mathbf{y} \quad (25)$$

or

$$\frac{\partial p_n}{\partial \beta_k} = \frac{e^{\mu_n T + \sigma_n^\top \beta}}{P_0^2 \sqrt{(T-t)}} \mathbb{E}_{(\mathbf{x}, \mathbf{y})} \left[u'(F(\mathbf{x})) u'(F(\mathbf{y})) (y_k - x_k) e^{\sqrt{T-t}\sigma_n^\top \mathbf{y}} \right] \quad (26)$$

Conditional on the filtration \mathcal{F}_t , $\frac{\partial p_n}{\partial \beta_k}$ is directly proportional to the $2K$ -dimensional integral on the right-hand side of (25). Unfortunately, this integral cannot be calculated analytically for general specifications of the functions $u(\cdot)$ and $\rho(\cdot)$. Nevertheless, its integrand exhibits a symmetry with respect to the variables of integration that allows us to use Lemma A.4. There are two cases to consider.

If $\sigma_n = \mathbf{0}$, the integral in (25) reads $\int_{\mathbb{R}^{2K}} g(\mathbf{x}, \mathbf{y}) f(\mathbf{x}, \mathbf{y}) d\mathbf{x} d\mathbf{y}$ with $g : \mathbb{R}^{2K} \rightarrow \mathbb{R}_{++}$ and $f : \mathbb{R}^{2K} \rightarrow \mathbb{R}$ defined by

$$\begin{aligned} g(\mathbf{x}, \mathbf{y}) &= u'(F(\mathbf{x})) u'(F(\mathbf{y})) e^{-\frac{\mathbf{y}^\top \mathbf{y} + \mathbf{x}^\top \mathbf{x}}{2}} \\ f(\mathbf{x}, \mathbf{y}) &= \mathbf{e}_k^\top (\mathbf{y} - \mathbf{x}) \end{aligned}$$

Since g is symmetric while $g(\mathbf{x}, \mathbf{y}) [f(\mathbf{x}, \mathbf{y}) + f(\mathbf{y}, \mathbf{x})] = 0 \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^K$, Lemma A.4 requires the integral to be zero.

For $\sigma_n \neq \mathbf{0}$, observe that the quantity multiplying $\frac{\partial p_n}{\partial \beta_k}$ on the left-hand side of (25) is invariant

with respect to $k \in \{1, \dots, K\}$. Summing over k , therefore, gives

$$\begin{aligned}
& \frac{\sqrt{T-t} (2\pi)^K P_0^2}{e^{\mu_n T + \sigma_n^\top (\beta + \frac{T-t}{2}\sigma)}} \sum_{k=1}^K \sigma_{nk} \frac{\partial p_n}{\partial \beta_k} \\
&= \int_{\mathbb{R}^{2K}} u'(F(\mathbf{y})) u'(F(\mathbf{x})) e^{\sqrt{T-t}\sigma_n^\top \mathbf{y} - \frac{\mathbf{y}^\top \mathbf{y} + \mathbf{x}^\top \mathbf{x}}{2}} \sum_{k=1}^K \sigma_{nk} (y_k - x_k) \, d\mathbf{x} d\mathbf{y} \\
&= \int_{\mathbb{R}^{2K}} g(\mathbf{x}, \mathbf{y}) h(\mathbf{x}, \mathbf{y}) \, d\mathbf{x} d\mathbf{y}
\end{aligned}$$

with g as before and $h : \mathbb{R}^{2K} \rightarrow \mathbb{R}$ given by $h(\mathbf{x}, \mathbf{y}) = \sigma_n^\top (\mathbf{y} - \mathbf{x}) e^{\sqrt{T-t}\sigma_n^\top \mathbf{y}}$. By Lemma A.4, the integral is strictly positive now since

$$\begin{aligned}
h(\mathbf{x}, \mathbf{y}) + h(\mathbf{y}, \mathbf{x}) &= \sigma_n^\top (\mathbf{y} - \mathbf{x}) \left(e^{\sqrt{T-t}\sigma_n^\top \mathbf{y}} - e^{\sqrt{T-t}\sigma_n^\top \mathbf{x}} \right) \\
&= e^{\sqrt{T-t}\sigma_n^\top \mathbf{x}} \sigma_n^\top (\mathbf{y} - \mathbf{x}) \left(e^{\sqrt{T-t}\sigma_n^\top (\mathbf{y} - \mathbf{x})} - 1 \right) \geq 0 \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^K
\end{aligned}$$

with the inequality strict on all of \mathbb{R}^{2K} except for the zero-measure subset which consists of the vectors (\mathbf{x}, \mathbf{y}) : $\sigma_n^\top (\mathbf{y} - \mathbf{x}) = 0$.

Supplementary Note for Theorem 2.1

I will demonstrate briefly how Lemma A.1 can be applied in the opening section of the preceding proof. For any realization of $(\beta, \mathbf{x}) \in \mathbb{R}^K \times \mathbb{R}^K$, we have

$$\begin{aligned}
& \frac{\partial}{\partial \beta_k} \left(u'(F(\mathbf{x})) e^{\mu_n T + \sigma_n^\top (\beta + \sqrt{T-t}\mathbf{x})} \right) \\
&= e^{\mu_n T + \sigma_n^\top (\beta + \sqrt{T-t}\mathbf{x})} \left(\sigma_{nk} u'(F(\mathbf{x})) + u''(F(\mathbf{x})) \frac{\partial F(\mathbf{x})}{\partial \beta_k} \right)
\end{aligned}$$

and

$$\begin{aligned}
& \frac{\partial^2}{\partial \beta_k^2} \left(u'(F(\mathbf{x})) e^{\mu_n T + \sigma_n^\top (\beta + \sqrt{T-t}\mathbf{x})} \right) \\
&= e^{\mu_n T + \sigma_n^\top (\beta + \sqrt{T-t}\mathbf{x})} \left(\begin{aligned} & \sigma_{nk}^2 u'(F(\mathbf{x})) \\ & + u''(F(\mathbf{x})) \left(2\sigma_{nk} \frac{\partial F(\mathbf{x})}{\partial \beta_k} + \frac{\partial^2 F(\mathbf{x})}{\partial \beta_k^2} \right) \\ & + u'''(F(\mathbf{x})) \left(\frac{\partial F(\mathbf{x})}{\partial \beta_k} \right)^2 \end{aligned} \right)
\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{\partial}{\partial \beta_k} u'(F(\mathbf{x})) &= u''(F(\mathbf{x})) \frac{\partial F(\mathbf{x})}{\partial \beta_k} \\ \frac{\partial^2}{\partial \beta_k^2} u'(F(\mathbf{x})) &= u''(F(\mathbf{x})) \frac{\partial^2 F(\mathbf{x})}{\partial \beta_k^2} + u'''(F(\mathbf{x})) \left(\frac{\partial F(\mathbf{x})}{\partial \beta_k} \right)^2\end{aligned}$$

while, by (23),

$$\frac{\partial^2 F(\mathbf{x})}{\partial \beta_k^2} = \frac{\partial^2 \rho(\beta + \sqrt{T-t}\mathbf{x})}{\partial \beta_k^2} + \sum_{i=1}^J \sigma_{ik}^2 e^{\mu_i T + \sigma_i^\top (\beta + \sqrt{T-t}\mathbf{x})}$$

Given $\beta_{-k} \in \mathbb{R}^{K-1}$, consider $F(\mathbf{x})$ as a function of only β_k and \mathbf{x}

$$F(\beta_k, \mathbf{x}) = \rho\left((\beta_k, \beta_{-k}) + \sqrt{T-t}\mathbf{x}\right) + \sum_{i=1}^J e^{\mu_i T + \sigma_i^\top ((\beta_k, \beta_{-k}) + \sqrt{T-t}\mathbf{x})}$$

and define also the function $H : \mathbb{R} \times \mathbb{R}^K \rightarrow \mathbb{R}_{++}$ by

$$H(\beta_k, \mathbf{x}) = u'(F(\beta_k, \mathbf{x})) e^{\mu_n T + \sigma_n^\top ((\beta_k, \beta_{-k}) + \sqrt{T-t}\mathbf{x})}$$

For the utility functions $u(\cdot)$ that are generally of interest in continuous-time finance, $H(\cdot)$ does satisfy the requirements of the lemma. \square

For the remaining of the section, keep in mind (25). The derivative of interest has the same sign as the quantity

$$\begin{aligned}\delta_{nk} &= e^{-\frac{(T-1)\sigma_n^\top \sigma_n}{2}} \int_{\mathbb{R}^{2K}} u'(F(\mathbf{y})) (y_k - x_k) u'(F(\mathbf{x})) e^{\sqrt{T-t}\sigma_n^\top \mathbf{y}} d\Phi(\mathbf{x}, \mathbf{y}) \\ &= \int_{\mathbb{R}^{2K}} u'\left(F\left(\mathbf{y} + \sqrt{T-t}\sigma_n\right)\right) \left(y_k + \sqrt{T-t}\sigma_{nk} - x_k\right) u'(F(\mathbf{x})) d\Phi(\mathbf{x}, \mathbf{y})\end{aligned}$$

the second equality applying Lemma A.2. When $\sigma_{nk} = 0$, this reads

$$\delta_{nk}^* = \int_{\mathbb{R}^{2K}} u'\left(F\left(\mathbf{y} + \sqrt{T-t}\sigma_n\right)\right) (y_k - x_k) u'(F(\mathbf{x})) d\Phi(\mathbf{x}, \mathbf{y})$$

And if, in addition, $\sigma_n = \sigma_{nm}\mathbf{e}_m$ for some $m \neq k$, it further simplifies to

$$\delta_{nk}^{**} = \int_{\mathbb{R}^{2K}} u'\left(F\left(\mathbf{y} + \sqrt{T-t}\sigma_{nm}\mathbf{e}_m\right)\right) (y_k - x_k) u'(F(\mathbf{x})) d\Phi(\mathbf{x}, \mathbf{y})$$

Proposition 3.1.1

Observe first that the terminal wealth specification in (13) can be expressed as $W(\mathcal{I}(\omega, t), \mathbf{x}) = W_1(\mathcal{I}(\omega, t), \mathbf{x}_{-k}) + W_2(\mathcal{I}(\omega, t), x_k)$ for some continuous functions $W_1 : \mathbb{R}^{K-1} \mapsto \mathbb{R}_{++}$ and $W_2 : \mathbb{R} \mapsto \mathbb{R}_{++}$.³⁰ For any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^K$, we have

$$\begin{aligned} u'(W(\mathbf{x})) &= \alpha\gamma e^{\alpha[W_1(\mathbf{x}_{-k}) + W_2(x_k)]} \\ u'\left(W\left(\mathbf{y} + \sqrt{T-t}\sigma_{nm}\mathbf{e}_m\right)\right) &= \alpha\gamma e^{\alpha[W_1(\mathbf{y}_{-k} + \sqrt{T-t}\sigma_{nm}\mathbf{e}_m) + W_2(y_k)]} \end{aligned}$$

with \mathbf{e}_m being the vector in \mathbb{R}^{K-1} now that has 1 as its m th entry and zeros elsewhere. Hence,

$$\frac{\delta_{nk}^{**}}{\alpha^2\gamma^2} = \int_{\mathbb{R}^{2(K-1)}} \left(\int_{\mathbb{R}^2} (gf)(x_k, y_k) d\Phi(x_k) d\Phi(y_k) \right) h(\mathbf{x}_{-k}, \mathbf{y}_{-k}) d\Phi(\mathbf{x}_{-k}) d\Phi(\mathbf{y}_{-k})$$

where $f : \mathbb{R}^2 \mapsto \mathbb{R}$, $g : \mathbb{R}^2 \mapsto \mathbb{R}_{++}$, and $h : \mathbb{R}^{2(K-1)} \mapsto \mathbb{R}_{++}$ are given by

$$\begin{aligned} f(x_k, y_k) &= y_k - x_k \\ g(x_k, y_k) &= e^{\alpha[W_2(x_k) + F_k(y_k)]} \\ h(\mathbf{x}_{-k}, \mathbf{y}_{-k}) &= e^{\alpha[W_1(\mathbf{y}_{-k} + \sqrt{T-t}\sigma_{nm}\mathbf{e}_m) + W_1(\mathbf{x}_{-k})]} \end{aligned}$$

By Lemma A.4, though, the two-dimensional integral in the brackets is zero and we are done.

Let us turn now to the terminal wealth specification in (14), which can be written as $W(\mathcal{I}(\omega, t), \mathbf{x}) = W_1(\mathcal{I}(\omega, t), \mathbf{x}_{-m}) + W_2(\mathcal{I}(\omega, t), x_m)$ for some continuous functions $W_1 : \mathbb{R}^{K-1} \mapsto \mathbb{R}_{++}$ and $W_2 : \mathbb{R} \mapsto \mathbb{R}_{++}$. For any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^K$, we get

$$\begin{aligned} u'(W(\mathbf{x})) &= \alpha\gamma e^{\alpha[W_1(\mathbf{x}_{-m}) + W_2(x_m)]} \\ u'\left(W\left(\mathbf{x} + \sqrt{T-t}\sigma_{nm}\mathbf{e}_m\right)\right) &= \alpha\gamma e^{\alpha[W_1(\mathbf{y}_{-m}) + W_2(y_m + \sqrt{T-t}\sigma_{nm})]} \end{aligned}$$

Hence,

$$\frac{\delta_{nk}^{**}}{\alpha^2\gamma^2} = \int_{\mathbb{R}^2} h(x_m, y_m) \left(\int_{\mathbb{R}^{2(K-1)}} (gf)(\mathbf{x}_{-m}, \mathbf{y}_{-m}) d\Phi(\mathbf{x}_{-m}) d\Phi(\mathbf{y}_{-m}) \right) d\Phi(x_m) d\Phi(y_m)$$

³⁰As I have pointed out already at the beginning of this appendix, everything here applies for an arbitrary realization of the process $\mathcal{I}(\omega, t)$. Given this, to minimize notational clutter, I am suppressing this argument whenever I use the terminal wealth functionals in what follows.

with $f : \mathbb{R}^{2(K-1)} \mapsto \mathbb{R}_{++}$, $g : \mathbb{R}^{2(K-1)} \mapsto \mathbb{R}$, and $h : \mathbb{R}^2 \mapsto \mathbb{R}_{++}$ now given by

$$\begin{aligned} g(\mathbf{x}_{-m}, \mathbf{y}_{-m}) &= e^{\alpha[W_1(\mathbf{x}_{-m}) + W_1(\mathbf{y}_{-m})]} \\ f(\mathbf{x}_{-m}, \mathbf{y}_{-m}) &= (\mathbf{y} - \mathbf{x}) \mathbf{e}_k = y_k - x_k \\ h(x_m, y_m) &= e^{\alpha[W_2(x_m) + W_2(y_m + \sqrt{T-t}\sigma_{nm})]} \end{aligned}$$

Yet, again by Lemma A.4, the $2(K-1)$ -dimensional integral in the brackets is zero. \square

To complete the analytical arguments that support the discussion in Section 3.1, notice the following. Under the specification in (13), equation (3) reads

$$\begin{aligned} p_n &= e^{\mu_m T + \sigma_{nm}(\beta_m + \frac{(T-t)\sigma_{nm}}{2})} \\ &\quad \frac{\mathbb{E}_{x_k} [e^{\alpha W_2(x_k)}] \mathbb{E}_{\mathbf{x}_{-k}} [e^{\alpha W_1(\mathbf{x}_{-k} + \sqrt{T-t}\sigma_{nm} \mathbf{e}_m)}]}{\mathbb{E}_{x_k} [e^{\alpha W_2(x_k)}] \mathbb{E}_{\mathbf{x}_{-k}} [e^{\alpha W_1(\mathbf{x}_{-k})}]} \\ &= e^{\mu_m T + \sigma_{nm}(\beta_m + \frac{(T-t)\sigma_{nm}}{2})} \frac{\mathbb{E}_{\mathbf{x}_{-k}} [e^{\alpha W_1(\mathbf{x}_{-k} + \sqrt{T-t}\sigma_{nm} \mathbf{e}_m)}]}{\mathbb{E}_{\mathbf{x}_{-k}} [e^{\alpha W_1(\mathbf{x}_{-k})}]} \end{aligned} \quad (27)$$

Under (??), on the other hand, (3) gives

$$\begin{aligned} p_n &= e^{\mu_m T + \sigma_{nm}(\beta_m + \frac{(T-t)\sigma_{nm}}{2})} \frac{\mathbb{E}_{x_m} [e^{\alpha W_2(x_m + \sqrt{T-t}\sigma_{nm})}] \mathbb{E}_{\mathbf{x}_{-m}} [e^{\alpha W_1(\mathbf{x}_{-m})}]}{\mathbb{E}_{x_m} [e^{\alpha W_2(x_m)}] \mathbb{E}_{\mathbf{x}_{-m}} [e^{\alpha W_1(\mathbf{x}_{-m})}]} \\ &= e^{\mu_m T + \sigma_{nm}(\beta_m + \frac{(T-t)\sigma_{nm}}{2})} \frac{\mathbb{E}_{x_m} [e^{\alpha W_2(x_m + \sqrt{T-t}\sigma_{nm})}]}{\mathbb{E}_{x_m} [e^{\alpha W_2(x_m)}]} \end{aligned} \quad (28)$$

Finally, the wealth specification in (15) is a special case of (14) given as $W(\mathcal{I}(\omega, t), \mathbf{x}) = \sum_{i=1}^K W_i(\mathcal{I}(\omega, t), x_i)$ for some continuous functions $W_i : \mathbb{R} \mapsto \mathbb{R}_{++}$. Clearly, $\mathbb{E}_{\mathbf{x}_{-m}} [e^{\alpha W_1(\mathbf{x}_{-m})}] = \prod_{i \neq m} \mathbb{E}_{x_i} [e^{\alpha W_i(x_i)}]$ now in (28) which, setting $m = n$, reads

$$p_n = e^{\mu_n T + \sigma_{nn}(\beta_n + \frac{(T-t)\sigma_{nn}}{2})} \frac{\mathbb{E}_{x_n} [e^{\alpha W_n(x_n + \sqrt{T-t}\sigma_{nn})}]}{\mathbb{E}_{x_n} [e^{\alpha W_n(x_n)}]} \quad (29)$$

Equations (17) and (21)

For (17), notice that, by renaming the variables of integration, we can re-write δ_{nk}^{**} as

$$\begin{aligned}\delta_{nk}^{**} &= \int_{\mathbb{R}^{2K}} \begin{bmatrix} u'(W(\mathbf{y} + \sqrt{T-t}\sigma_{nm}\mathbf{e}_m)) u'(W(\mathbf{x})) y_k \\ -u'(W(\mathbf{x} + \sqrt{T-t}\sigma_{nm}\mathbf{e}_m)) u'(W(\mathbf{y})) y_k \end{bmatrix} d\Phi(\mathbf{x}, \mathbf{y}) \\ &= \mathbb{E}_{y_k} \left[\mathbb{E}_{(\mathbf{x}, \mathbf{y}-k)} \left[\begin{bmatrix} u'(W(\mathbf{y} + \sqrt{T-t}\sigma_{nm}\mathbf{e}_m)) u'(W(\mathbf{x})) \\ -u'(W(\mathbf{x} + \sqrt{T-t}\sigma_{nm}\mathbf{e}_m)) u'(W(\mathbf{y})) \end{bmatrix} y_k \right] \right]\end{aligned}$$

The claim follows since $\mathbb{E}_{y_k}[y_k] = 0$. Regarding (21), as long as $\sigma_{nk} = 0$, the same argument applies on δ_{nk}^* once $\sigma_{nm}\mathbf{e}_m$ is replaced by σ_n .

Propositions 3.1.2 and 3.2.1

First, I will establish Proposition 3.2.1. From this, Proposition 3.1.2 follows more or less immediately.

Recall that the collection of indices $K_n = \{m \in \{1, \dots, K\} : \sigma_{nm} \neq 0\}$ denotes those components of the Brownian vector that do affect the terminal dividend of the n th security. By contrast, $N_k = \{n' \in \{1, \dots, N\} : \sigma_{n'k} \neq 0\}$ is the collection of risky securities whose terminal dividends vary with the k th Brownian motion. Obviously, as $\sigma_{nk} = 0$, $n \notin N_k$ while $M \equiv |K_n| < K$.

Notice also that, by permuting if necessary the elements of the index set $\{1, \dots, K\}$, it is without any loss of generality to take the first M of these indices as the set K_n and the last index to correspond to the k th dimension, the one under study. In what follows, a vector such as \mathbf{x}_M lives in the space \mathbb{R}^M , depicting a collection $\{\beta_m(\omega, T) - \beta_m(\omega, t)\}_{m \in K_n}$ of realizations for the increments of these M Brownian motions. Similarly, albeit with a slight abuse of notation, a vector such as \mathbf{x}_{-M} is to be found in \mathbb{R}^{K-M-1} and represents a collection $\{\beta_{k'}(\omega, T) - \beta_{k'}(\omega, t)\}_{k' \notin (K_n \cup \{k\})}$ of realizations for the increments of the Brownian motions that are listed, under the new indexing, from $M+1$ to $K-1$. Finally, a vector such as $\mathbf{x}_{-(M,k)}$ is in \mathbb{R}^{K-M} and refers to a collection of realizations for the increments of those Brownian motions that do not belong to the set $K_n \cup \{k\}$.

Step 1. Observe that

$$\mathbb{E}_{(\mathbf{x}, \mathbf{y})} [u'(W(\mathbf{y})) (y_k - x_k) u'(W(\mathbf{x}))] = 0 \tag{30}$$

which, by renaming the variables of integration $\mathbf{y}_M \in \mathbb{R}^M$, can be re-written as

$$\mathbb{E}_{\mathbf{y}_{-(M,k)}} \left[\mathbb{E}_{(\mathbf{z}_M, y_k)} [u'(W(\mathbf{y}_{-M}, \mathbf{z}_M))] \mathbb{E}_{\mathbf{x}} [(y_k - x_k) u'(W(\mathbf{x})) | y_k] | \mathbf{y}_{-M}] \right] = 0$$

Hence,

$$\begin{aligned} \delta_{nk}^* &= \mathbb{E}_{(\mathbf{x}, \mathbf{y})} \left[u' \left(W \left(\mathbf{y} + \sqrt{T-t} \sigma_n \right) \right) (y_k - x_k) u' (W(\mathbf{x})) \right] \\ &= \mathbb{E}_{(\mathbf{x}, \mathbf{y})} \left[u' \left(W \left(\mathbf{y} + \sqrt{T-t} \sigma_n \right) \right) (y_k - x_k) u' (W(\mathbf{x})) \right] \\ &\quad - \mathbb{E}_{\mathbf{y}_{-(M,k)}} \left[\mathbb{E}_{(\mathbf{z}_M, y_k)} [u'(W(\mathbf{y}_{-M}, \mathbf{z}_M))] \mathbb{E}_{\mathbf{x}} [(y_k - x_k) u'(W(\mathbf{x})) | y_k] | \mathbf{y}_{-(M,k)}] \right] \\ &= \mathbb{E}_{\mathbf{y}_{-(M,k)}} \left[\mathbb{E}_{y_k} \left[\left(\begin{array}{c} \mathbb{E}_{\mathbf{y}_M} [u'(W(\mathbf{y} + \sqrt{T-t} \sigma_n)) | \mathbf{y}_{-(M,k)}] \\ - \mathbb{E}_{\mathbf{z}_M} [u'(W(\mathbf{y}_{-M}, \mathbf{z}_M)) | \mathbf{y}_{-(M,k)}] \end{array} \right) \mathbb{E}_{\mathbf{x}} [(y_k - x_k) u'(W(\mathbf{x})) | y_k] \right] \right] \\ &= \mathbb{E}_{\mathbf{y}_{-(M,k)}} \left[\mathbb{E}_{y_k} \left[\left(\begin{array}{c} e^{-\frac{(T-t)\sigma_n^T \sigma_n}{2}} \mathbb{E}_{\mathbf{y}_M} [u'(W(\mathbf{y})) e^{\sqrt{T-t} \sigma_n^T \mathbf{y}} | \mathbf{y}_{-(M,k)}] \\ - \mathbb{E}_{\mathbf{z}_M} [u'(W(\mathbf{y}_{-M}, \mathbf{z}_M)) | \mathbf{y}_{-(M,k)}] \end{array} \right) \right] \right] \\ &= \mathbb{E}_{\mathbf{y}_{-(M,k)}} \left[\mathbb{E}_{y_k} \left[\left(\begin{array}{c} \left(\frac{e^{-\frac{(T-t)\sigma_n^T \sigma_n}{2}} \mathbb{E}_{\mathbf{y}_M} [u'(W(\mathbf{y})) e^{\sqrt{T-t} \sigma_n^T \mathbf{y}} | \mathbf{y}_{-(M,k)}]}{\mathbb{E}_{\mathbf{z}_M} [u'(W(\mathbf{y}_{-M}, \mathbf{z}_M)) | \mathbf{y}_{-(M,k)}]} - 1 \right) \\ \mathbb{E}_{\mathbf{z}_M} [u'(W(\mathbf{y}_{-M}, \mathbf{z}_M)) | \mathbf{y}_{-(M,k)}] \mathbb{E}_{\mathbf{x}} [(y_k - x_k) u'(W(\mathbf{x})) | y_k] \end{array} \right) \right] \right] \end{aligned}$$

the last equality following from Lemma A.2 (and the fact that \mathbf{y}_M lists exhaustively the Brownian dimensions that affect the n th terminal dividend).

Step 2. Fix now an arbitrary point $\mathbf{y}_{-(M,k)} \in \mathbb{R}^{K-M-1}$. I will show that the function $g_1 : \mathbb{R} \mapsto \mathbb{R}$ given by XXXXX

$$g_1(y_k) = \frac{e^{-\frac{(T-t)\sigma_n^T \sigma_n}{2}} \mathbb{E}_{\mathbf{y}_M} [u'(W(\mathbf{y})) e^{\sqrt{T-t} \sigma_n^T \mathbf{y}}]}{\mathbb{E}_{\mathbf{z}_M} [u'(W(\mathbf{y}_{-M}, \mathbf{z}_M))]} - 1$$

is monotone under the conditions of the proposition.

To this end, fix an arbitrary $y_k \in \mathbb{R}$. Since $\sigma_{nk} = 0$, $g_1'(y_k)$ has the same sign as the quantity

$$\begin{aligned} I(y_k) &= e^{-\frac{(T-t)\sigma_n^T \sigma_n}{2}} \mathbb{E}_{(\mathbf{y}_M, \mathbf{z}_M)} \left[\left(\begin{array}{c} u''(W(\mathbf{y})) u'(W(\mathbf{y}_{-M}, \mathbf{z}_M)) \frac{\partial W(\mathbf{y})}{\partial y_k} \\ - u'(W(\mathbf{y})) u''(W(\mathbf{y}_{-M}, \mathbf{z}_M)) \frac{\partial W(\mathbf{y}_{-M}, \mathbf{z}_M)}{\partial y_k} \end{array} \right) e^{\sqrt{T-t} \sigma_n^T \mathbf{y}} \right] \\ &= r_A e^{-\frac{(T-t)\sigma_n^T \sigma_n}{2}} \\ &\quad \mathbb{E}_{(\mathbf{y}_M, \mathbf{z}_M)} \left[u'(W(\mathbf{y})) u'(W(\mathbf{y}_{-M}, \mathbf{z}_M)) \left(\frac{\partial W(\mathbf{y}_{-M}, \mathbf{z}_M)}{\partial y_k} - \frac{\partial W(\mathbf{y})}{\partial y_k} \right) e^{\sqrt{T-t} \sigma_n^T \mathbf{y}} \right] \end{aligned}$$

Under condition (i) of the proposition, though, we have

$$\begin{aligned} W(\mathbf{y}) &= \rho(\mathbf{y}_{-k}) + \sum_{n'=1}^N D_{n'}(\mathbf{y}) \\ &= \rho(\mathbf{y}_{-k}) + \sum_{n'=1}^N e^{\mu_{n'}T + \sigma_{n'}^T \beta + \sqrt{T-t}(\sum_{k' \notin K_n} \sigma_{n'k'} y_{k'} + \sum_{m \in K_n} \sigma_{n'm} y_m)} \end{aligned}$$

i.e.

$$\frac{\partial W(\mathbf{y})}{\partial y_k} = \sqrt{T-t} \sum_{n' \in N_k} \sigma_{n'k} e^{\mu_{n'}T + \sigma_{n'}^T \beta + \sqrt{T-t}(\sum_{k' \notin K_n} \sigma_{n'k'} y_{k'} + \sum_{m \in K_n} \sigma_{n'm} y_m)}$$

Hence,

$$\begin{aligned} I(y_k) &= r_A \sqrt{T-t} e^{-\frac{(T-t)\sigma_n^T \sigma_n}{2}} \\ &\sum_{n' \in N_k} \sigma_{n'k} e^{\mu_{n'}T + \sigma_{n'}^T \beta + \sqrt{T-t} \sum_{k' \notin K_n} \sigma_{n'k'} y_{k'}} \mathbb{E}_{(\mathbf{y}_M, \mathbf{z}_M)} [u'(W(\mathbf{y})) u'(W(\mathbf{y}_{-M}, \mathbf{z}_M)) h_{n'}(\mathbf{y}_M, \mathbf{z}_M)] \end{aligned}$$

with $h_{n'} : \mathbb{R}^{2M} \mapsto \mathbb{R}$ defined as

$$\begin{aligned} h_{n'}(\mathbf{y}_M, \mathbf{z}_M) &= \left(e^{\sqrt{T-t} \sum_{m \in K_n} \sigma_{n'm} z_m} - e^{\sqrt{T-t} \sum_{m \in K_n} \sigma_{n'm} y_m} \right) e^{\sqrt{T-t} \sigma_n^T \mathbf{y}} \\ &= e^{\sqrt{T-t} \left(\sum_{m' \in K_n \setminus (K_n \cap K_{n'})} \sigma_{nm'} y_{m'} + \sum_{m \in K_n \cap K_{n'}} \sigma_{nm} y_m \right)} \\ &\quad \left(e^{\sqrt{T-t} \sum_{m \in K_n \cap K_{n'}} \sigma_{n'm} z_m} - e^{\sqrt{T-t} \sum_{m \in K_n \cap K_{n'}} \sigma_{n'm} y_m} \right) \end{aligned}$$

where the second equality deploys the fact that $\sigma_{n'm} = 0 \forall m \notin K_{n'}$. But, under condition (iv) of the proposition, $\sigma_{n'm} = \lambda_{n'} \sigma_{nm} \forall m \in K_n \cap K_{n'}$. Therefore,

$$\begin{aligned} h_{n'}(\mathbf{y}_M, \mathbf{z}_M) &= e^{\sqrt{T-t} \left(\sum_{m' \in K_n \setminus (K_n \cap K_{n'})} \sigma_{nm'} y_{m'} + \sum_{m \in K_n \cap K_{n'}} \sigma_{nm} y_m \right)} \\ &\quad \left(e^{\lambda_{n'} \sqrt{T-t} \sum_{m \in K_n \cap K_{n'}} \sigma_{nm} z_m} - e^{\lambda_{n'} \sqrt{T-t} \sum_{m \in K_n \cap K_{n'}} \sigma_{nm} y_m} \right) \end{aligned}$$

If $\lambda_{n'} > 0$ ($\lambda_{n'} < 0$), then $h_{n'}(\mathbf{y}_M, \mathbf{z}_M) + h_{n'}(\mathbf{z}_M, \mathbf{y}_M) \leq 0$ (≥ 0) on \mathbb{R}^{2M} with equality only on the zero-measure subset consisting of the vectors $(\mathbf{y}_M, \mathbf{z}_M) : \sum_{m \in K_n \cap K_{n'}} \sigma_{nm} (y_m - z_m) = 0$. By Lemma A.4, therefore, the expectation in the i th term of the sum above is negative (positive) if $\lambda_{n'} > 0$ ($\lambda_{n'} < 0$).³¹ Equivalently, the i th term itself is negative (positive) if

³¹To use the lemma here, let $g := h_{n'}$ and define $f : \mathbb{R}^{2M} \mapsto \mathbb{R}_{++}$ by $f(\mathbf{y}_M, \mathbf{z}_M) = u'(W(\mathbf{y})) u'(W(\mathbf{y}_{-M}, \mathbf{z}_M)) e^{-\frac{\mathbf{y}_M^T \mathbf{y}_M + \mathbf{z}_M^T \mathbf{z}_M}{2}}$.

$\lambda_{n'}\sigma_{n'k} > 0$ ($\lambda_{n'}\sigma_{n'k} < 0$).

To sign the entire sum, it suffices that all of its terms are of the same sign. And this is guaranteed by condition (v) of the proposition. To see this, consider the collection $\cup_{m \in K_n} N_m$ of those risky securities whose terminal dividend varies with at least one of the Brownian components that affect the n th dividend. Condition (iv) required a proportionality constant $\lambda_{n'}$ only for those securities that are simultaneously members of this collection and of N_k ; that is, for the members of the collection $n' \in \cup_{m \in K_n} (N_m \cap N_k)$. If $\lambda_{n'}\sigma_{n'k}$ maintains, therefore, the same sign on this set, $I(y_k)$ will have the opposite sign.

Step 3. Define the function $g_2 : \mathbb{R} \mapsto \mathbb{R}$ by

$$g_2(y_k) = \mathbb{E}_{\mathbf{z}_M} [u'(W(\mathbf{y}_{-M}, \mathbf{z}_M))] \mathbb{E}_{\mathbf{x}} [(y_k - x_k) u'(W(\mathbf{x}))]$$

Since $u'(\cdot) > 0$, Lemma A.5 in Appendix A ensures the existence of some $y_k^0 \in \mathbb{R}$ with $(y_k - y_k^0) g(y_k) > 0 \forall y_k \in \mathbb{R} \setminus \{y_k^0\}$.

Step 4. Let $\lambda_{n'}\sigma_{n'k} > 0$. By Step 2, $\lambda_{n'}\sigma_{n'k}g_1(\cdot)$ is strictly decreasing on \mathbb{R} . But then

$$\begin{aligned} & \mathbb{E}_{y_k} [\lambda_{n'}\sigma_{n'k}g_1(y_k) g_2(y_k)] \\ & < \int_{y_k \in (y_k^0, +\infty)} \lambda_{n'}\sigma_{n'k}g_1(y_k^0) g_2(y_k) d\Phi(y_k) + \int_{y_k \in (-\infty, y_k^0)} \lambda_{n'}\sigma_{n'k}g_1(y_k^0) g_2(y_k) d\Phi(y_k) \\ & = \lambda_{n'}\sigma_{n'k}g_1(y_k^0) \mathbb{E}_{y_k} [g_2(y_k)] \end{aligned}$$

and, thus,

$$\lambda_{n'}\sigma_{n'k}\delta_{nk}^* = \mathbb{E}_{\mathbf{y}_{-(M,k)}} [\mathbb{E}_{y_k} [\lambda_{n'}\sigma_{n'k}g_1(y_k) g(y_k)]] < \lambda_{n'}\sigma_{n'k}g_1(y_k^0) \mathbb{E}_{\mathbf{y}_{-(M,k)}} [\mathbb{E}_{y_k} [g(y_k)]] = 0$$

the last equality following from (30). \square

Proposition 3.1.2 restricts K_n to be the singleton $\{m\}$. Now, any security $n' \in N_k \setminus (N_m \cap N_k)$ has $\sigma_{n'm} = 0$ so that the corresponding $h_{n'}$ is the zero function. Which means that the condition $\lambda_{n'}\sigma_{n'k}\lambda_{n''}\sigma_{n''k} > 0 \forall n', n'' \in \cup_{m \in K_n} (N_m \cap N_k)$ reads now $\sigma_{n'm}\sigma_{n'k}\sigma_{n''m}\sigma_{n''k} > 0 \forall n', n'' \in N_m \cap N_k$ (and becomes redundant if N_k is also a singleton as in Corollary 3.1.3). The preceding argument establishes that $\sigma_{nm}\sigma_{n'm}\sigma_{n'k}\delta_{nk}^* < 0$ with respect to any $n' \in N_m \cap N_k$.

Propositions 3.1.3 and 3.2.2

The proof of Proposition 3.2.2 proceeds in the same fashion as that of Proposition 3.2.1.

Step 1. Fixing an arbitrary point $\mathbf{y}_{-(M,k)} \in \mathbb{R}^{K-M-1}$, the function $g_1 : \mathbb{R} \mapsto \mathbb{R}_{--}$ is again

strictly monotone, exhibiting $\sigma_{n'k}g'(y_k) > 0 \forall y_k \in \mathbb{R}$ in this case.³²

To see this, observe that now

$$I(y_k) = e^{-\frac{(T-t)\sigma_n^T\sigma_n}{2}} \mathbb{E}_{(\mathbf{y}_M, \mathbf{z}_M)} \left[\begin{array}{c} u'(F(\mathbf{y})) u'(F(\mathbf{y}_{-M}, \mathbf{z}_M)) \\ \left[r_A(F(\mathbf{y}_{-M}, \mathbf{z}_M)) \frac{F(\mathbf{y}_{-M}, \mathbf{z}_M)}{\partial y_k} - r_A(F(\mathbf{y})) \frac{F(\mathbf{y})}{\partial y_k} \right] e^{\sqrt{T-t}\sigma_n^T \mathbf{y}} \end{array} \right]$$

Under condition (iii) of Proposition 3.2.2, the terminal-period endowment is a function $e(\mathbf{y}_{-(M,k)})$ while $\sigma_{n'm} = 0 \forall (n', m) \in N_k \times K_n$. Condition (iv), moreover, requires that $\sigma_{n'm} = \sigma_{nm}$ for any dividend n' with $\sigma_{n'm} \neq 0$ for some $m \in K_n$. Hence,³³

$$\begin{aligned} W(\mathbf{y}) &= e(\mathbf{y}_{-(M,k)}) + \sum_{n'=1}^N D_{n'}(\mathbf{y}) \\ &= e(\mathbf{y}_{-(M,k)}) + \sum_{n' \in \{1, \dots, N\} \setminus \cup_{m \in K_n} N_m} e^{\mu_{n'}T + \sigma_{n'}^T \beta + \sqrt{T-t} \sum_{k' \notin K_n} \sigma_{n'k'} y_{k'}} \\ &\quad + e^{\sqrt{T-t} \sum_{m \in K_n} \sigma_{nm} y_m} \sum_{n' \in \cup_{m \in K_n} N_m} e^{\mu_{n'}T + \sigma_{n'}^T \beta + \sqrt{T-t} \sum_{k' \notin K_n} \sigma_{n'k'} y_{k'}} \end{aligned}$$

and, thus,

$$\begin{aligned} \frac{\partial W(\mathbf{y})}{\partial y_k} &= \sqrt{T-t} \sum_{n' \in N_k} \sigma_{n'k} e^{\mu_{n'}T + \sigma_{n'}^T \beta + \sqrt{T-t} \sum_{k' \notin K_n} \sigma_{n'k'} y_{k'}} \\ &= \sqrt{T-t} \sum_{n' \in N_k} \sigma_{n'k} D_{n'}(\mathbf{y}_{-M}) \end{aligned}$$

That is,

$$\frac{e^{\frac{(T-t)\sigma_n^T\sigma_n}{2}} I(y_k)}{\sqrt{T-t}} = \mathbb{E}_{(\mathbf{y}_M, \mathbf{z}_M)} [u'(F(\mathbf{y})) u'(F(\mathbf{y}_{-M}, \mathbf{z}_M)) g(\mathbf{y}_{-M}, \mathbf{z}_M)] \sum_{n' \in N_k} \sigma_{n'k} D_{n'}(\mathbf{y}_{-M})$$

³²Recall the one before the last footnote. Under conditions (i)-(ii) of either of Propositions 3.1.3 and 3.2.2, $g_1(\cdot)$ is strictly negative everywhere on \mathbb{R} .

³³Some remarks about the way the terminal wealth is written out here. On the right-hand side of the second equality, I sum across the N terminal dividends by partitioning them into two sets. The first summation collects the ones that are *not* correlated with any of the Brownian dimensions that affect the dividend of the n th security. In the exponent of the typical term here, no terms of the form $\sigma_{n'm}y_m$ with $m \in K_n$ appear as they are all zero. The second summation collects the remaining dividends. In the exponent of the typical term now, there *are* terms of the form $\sigma_{n'm}y_m$ with $m \in K_n$. Yet, in all of them, $\sigma_{n'm} = \sigma_{nm}$ due to condition (iv). The product of the corresponding exponentials can be, therefore, pulled out of the summation. In the exponent of the typical term of the second summation, there can also be terms of the form $\sigma_{n'k'}y_{k'}$ with $k' \notin K_n$. The corresponding exponentials stay inside the summation. Observe finally that, by condition (iii), no dividend n' whose exponent includes the term $\sigma_{n'k}y_k$ is to be found in the second summation.

with $g : \mathbb{R}^{K+M} \mapsto \mathbb{R}$ s.t.

$$g(\mathbf{y}_{-M}, (\mathbf{y}_M, \mathbf{z}_M)) = [r_A(F(\mathbf{y}_{-M}, \mathbf{z}_M)) - r_A(F(\mathbf{y}))] e^{\sqrt{T-t} \sum_{m \in K_n} \sigma_{nm} y_m}$$

But

$$\begin{aligned} & g(\mathbf{y}_{-M}, (\mathbf{y}_M, \mathbf{z}_M)) + g(\mathbf{y}_{-M}, (\mathbf{z}_M, \mathbf{y}_M)) \\ = & [r_A(F(\mathbf{y}_{-M}, \mathbf{z}_M)) - r_A(F(\mathbf{y}))] \left(e^{\sqrt{T-t} \sum_{m \in K_n} \sigma_{nm} y_m} - e^{\sqrt{T-t} \sum_{m \in K_n} \sigma_{nm} z_m} \right) \end{aligned}$$

is non-negative on \mathbb{R}^{2M} , being zero only on the zero-measure set consisting of the vectors $(\mathbf{y}_M, \mathbf{z}_M) : \sum_{m \in K_n} \sigma_{nm} (y_m - z_m) = 0$.³⁴ This implies that the expectation above is positive (Lemma A.4), allowing in turn condition (v) of the proposition to ensure that $g'_1(y_k)$ has the same sign as the factor loading $\sigma_{n'k}$ of the dividend of any $n' \in N_k$.

Step 2. By the same argument as in the last two steps of the proof of Proposition 3.2.1, one can establish that $\sigma_{n'k} g_1(\cdot)$ is strictly increasing on \mathbb{R} only if $\sigma_{n'k} \delta_{n'k}^* > 0$. \square

For Proposition 3.1.3, let $K_n = \{m\}$. The requirements $\sigma_{n'm} = 0 \forall (n', m) \in N_k \times K_n$ and $\sigma_{n'm} = \sigma_{nm} \forall m \in K_n$ and $\forall n' \in \cup_{m \in K_n} N_m$ reduce now, respectively, to $N_k \cap N_m = \emptyset$ and $\sigma_{n'm} = \sigma_{nm} \forall n' \in N_m$. The result reads $\sigma_{n'k} \delta_{n'k}^{**} > 0$.

Proposition 3.2.3

Let the Brownian dimension k be arbitrary and $u(c) = \gamma c^\alpha$ ($\gamma, \alpha < 0$). By condition (iii) of the claim, we have $u'(F(\mathbf{x} + \sqrt{T-t} \sigma_n)) = \lambda_n^{\alpha-1} u'(F(\mathbf{x}))$. Hence, (24) now reads

$$\begin{aligned} \frac{\sqrt{T-t} P_0^2}{\lambda_n^{\alpha-1} e^{\mu_n T + \sigma_n^2 \left(\beta - \frac{(T-t)\sigma_n}{2} \right)}} \frac{\partial p_n}{\partial \beta_k} &= \mathbb{E}_{\mathbf{y}} \left[\left(y_k + \sqrt{T-t} \sigma_{nk} \right) u'(F(\mathbf{y})) \right] \mathbb{E}_{\mathbf{x}} [u'(F(\mathbf{x}))] \\ &\quad - \mathbb{E}_{\mathbf{x}} [x_k u'(F(\mathbf{x}))] \mathbb{E}_{\mathbf{y}} [u'(F(\mathbf{y}))] \\ &= \sqrt{T-t} \sigma_{nk} \mathbb{E}_{\mathbf{y}} [u'(F(\mathbf{y}))]^2 \\ &\quad + \mathbb{E}_{(\mathbf{x}, \mathbf{y})} [u'(F(\mathbf{x})) u'(F(\mathbf{y})) (y_k - x_k)] \\ &= \sqrt{T-t} \sigma_{nk} \mathbb{E}_{\mathbf{y}} [u'(F(\mathbf{y}))]^2 = \sqrt{T-t} \sigma_{nk} P_0^2 \end{aligned}$$

With $\alpha = 0$, this applies also when the utility function is logarithmic.

³⁴It is at this point of the proof that condition (iv) is deployed. The condition permitted the term $\eta = e^{\sqrt{T-t} \sum_{m \in K_n} \sigma_{nm} y_m}$ to be factored out of the second summation when the expression for the terminal wealth was written out above. By this, the condition ensures that $W(\mathbf{y})$ is strictly increasing in this term. Which, under DARA, implies in turn that $\frac{dr_A(W(\eta))}{d\eta} < 0$ and allows the signing of the quantity $g(\mathbf{y}_{-M}, (\mathbf{y}_M, \mathbf{z}_M)) + g(\mathbf{y}_{-M}, (\mathbf{z}_M, \mathbf{y}_M))$.

Theorem 4.1

Let $N = K$. As is well-known, in the presence of a money market account, dynamic completeness is equivalent to the dispersion matrix of the securities prices

$$\left[\frac{\partial P_n(\mathcal{I}(\omega, t))}{\partial \beta_k(\omega, t)} \right]_{(n,k) \in \{0, \dots, K\} \times \{1, \dots, K\}}$$

having almost everywhere rank equal to K , the number of the sources of uncertainty (see, for example, Sections 4.1-4.4 and Theorem 5.6 in Nielsen [31]). Here, one of the securities, the zero-coupon bond, is itself a money market account (the corresponding self-financing strategy being to hold $1/P_0(\mathcal{I}(\omega, t))$ units of the bond). Its price P_0 , moreover, is positive everywhere on $\Omega \times [0, T]$. Renormalizing, therefore, dynamic completeness is equivalent to the $K \times K$ dispersion matrix

$$J_p(\mathcal{I}(\omega, t)) = \left[\frac{\partial}{\partial \beta_k(\omega, t)} \left(\frac{P_n(\mathcal{I}(\omega, t))}{P_0(\mathcal{I}(\omega, t))} \right) \right]_{(n,k) \in \{1, \dots, K\} \times \{1, \dots, K\}}$$

having almost everywhere on $\Omega \times [0, T]$ rank equal to K , the dimension of the Brownian process. In what follows, I establish that the latter condition is equivalent to the matrix Σ being nonsingular. To this end, I will consider the n th row of the matrix $J_p(\mathcal{I}(\omega, t))$ as a vector and denote it by $\mathbf{j}_{p,n}$.

Only If. To establish the contrapositive statement, suppose that Σ is singular. There exists, therefore, $\mathbf{v} \in \mathbb{R}^K \setminus \{\mathbf{0}\}$ s.t. $\Sigma \mathbf{v} = \mathbf{0}$. Take now $\mathbf{a} \in \mathbb{R}^K \setminus \{\mathbf{0}\}$ s.t. $\mathbf{a}^\top \mathbf{v} \neq 0$ and consider the hyperplane $H_{\mathbf{a}} = \{\mathbf{x} \in \mathbb{R}^K : \mathbf{a}^\top \mathbf{x} = 0\}$. For an arbitrary $\mathbf{x}_0 \in H_{\mathbf{a}}$, consider also the line through \mathbf{x}_0 in the direction of \mathbf{v} , $L(\mathbf{x}_0; \mathbf{v}) = \{\mathbf{x} \in \mathbb{R}^K : \mathbf{x} = \mathbf{x}_0 + t\mathbf{v}, t \in \mathbb{R}\}$. Since $\mathbf{v} \not\parallel H_{\mathbf{a}}$, \mathbb{R}^K can be spanned as $\cup_{\mathbf{x}_0 \in H_{\mathbf{a}}} L(\mathbf{x}_0; \mathbf{v})$.³⁵ Hence, for the n th risky security,

$$\mathbf{a}^\top \mathbf{j}_{p,n} = \int_{\mathbb{R}^{2K}} u'(F(\mathbf{x})) u'(F(\mathbf{y})) e^{\sigma_n^\top \mathbf{y}} \mathbf{a}^\top (\mathbf{y} - \mathbf{x}) e^{-\frac{\mathbf{y}^\top \mathbf{y} + \mathbf{x}^\top \mathbf{x}}{2}} d\mathbf{x} d\mathbf{y} = \int_{H_{\mathbf{a}} \times H_{\mathbf{a}}} I(\mathbf{x}_0, \mathbf{y}_0; \mathbf{a}) d\mathbf{x}_0 d\mathbf{y}_0$$

where $I : H_{\mathbf{a}} \times H_{\mathbf{a}} \mapsto \mathbb{R}$ is given by

$$I(\mathbf{x}_0, \mathbf{y}_0; \mathbf{a}) = \int_{L(\mathbf{x}_0; \mathbf{v}) \times L(\mathbf{y}_0; \mathbf{v})} u'(F(\mathbf{x})) u'(F(\mathbf{y})) e^{\sigma_n^\top \mathbf{y}} \mathbf{a}^\top (\mathbf{y} - \mathbf{x}) e^{-\frac{\mathbf{y}^\top \mathbf{y} + \mathbf{x}^\top \mathbf{x}}{2}} d\mathbf{x} d\mathbf{y}$$

³⁵Let $\{\mathbf{v}_k\}_{k=1}^{K-1}$ be a basis for the hyperplane $H_{\mathbf{a}}$. As this basis and \mathbf{v} are not collinear, $\{\mathbf{v}, \mathbf{v}_1, \dots, \mathbf{v}_{K-1}\}$ is a basis of \mathbb{R}^K so that any $\mathbf{x} \in \mathbb{R}^K$ can be written uniquely as $\mathbf{x} = \sum_{k=1}^{K-1} t_k \mathbf{v}_k + t\mathbf{v}$ for some $t, t_1, \dots, t_{K-1} \in \mathbb{R}$. Equivalently, as $\mathbf{x} = \mathbf{x}_0 + t\mathbf{v}$ for a unique $\mathbf{x}_0 = \sum_{k=1}^{K-1} t_k \mathbf{v}_k \in H_{\mathbf{a}}$.

Observe now that, $\forall \mathbf{x} \in L(\mathbf{x}_0; \mathbf{v})$, $\mathbf{a}^\top \mathbf{x} = t\mathbf{a}^\top \mathbf{v}$ and $\sigma_n^\top \mathbf{x} = \sigma_n^\top \mathbf{x}_0 \forall n = 1, \dots, N$. The latter relation dictates that the terminal-period wealth is a function of \mathbf{x}_0 rather than \mathbf{x} on $L(\mathbf{x}_0; \mathbf{v})$. Therefore,

$$\begin{aligned}
& I(\mathbf{x}_0, \mathbf{y}_0; \mathbf{a}) \\
&= \int_{\mathbb{R}^2} u'(F(\mathbf{x}_0)) u'(F(\mathbf{y}_0)) e^{\sigma_n^\top \mathbf{y}_0} (t - \tau) \mathbf{a}^\top \mathbf{v} e^{-\frac{(\mathbf{y}_0 + t\mathbf{v})^\top (\mathbf{y}_0 + t\mathbf{v}) + (\mathbf{x}_0 + \tau\mathbf{v})^\top (\mathbf{x}_0 + \tau\mathbf{v})}{2}} dt d\tau \\
&= u'(F(\mathbf{x}_0)) u'(F(\mathbf{y}_0)) e^{\sigma_n^\top \mathbf{y}_0} \int_{\mathbb{R}^2} \mathbf{a}^\top (t - \tau) \mathbf{v} e^{-\frac{(\mathbf{y}_0 + t\mathbf{v})^\top (\mathbf{y}_0 + t\mathbf{v}) + (\mathbf{x}_0 + \tau\mathbf{v})^\top (\mathbf{x}_0 + \tau\mathbf{v})}{2}} dt d\tau \\
&= u'(F(\mathbf{x}_0)) u'(F(\mathbf{y}_0)) e^{\sigma_n^\top \mathbf{y}_0} \mathbb{E}_{(\mathbf{z}, \tilde{\mathbf{z}})} [\mathbf{a}^\top (\tilde{\mathbf{z}} - \mathbf{z})]
\end{aligned}$$

where $(\mathbf{z}, \tilde{\mathbf{z}}) \sim \mathcal{N}\left(-(\mathbf{x}_0, \mathbf{y}_0), \begin{pmatrix} \mathbf{v}\mathbf{v}^\top & \mathbf{0} \\ \mathbf{0} & \mathbf{v}\mathbf{v}^\top \end{pmatrix}\right)$. That is,

$$I(\mathbf{x}_0, \mathbf{y}_0; \mathbf{a}) = u'(F(\mathbf{x}_0)) u'(F(\mathbf{y}_0)) e^{\sigma_n^\top \mathbf{y}_0} \mathbf{a}^\top (\mathbf{x}_0 - \mathbf{y}_0) = 0 - 0$$

Given the arbitrary choice of security and node, $\mathbf{a}^\top \mathbf{j}_{p,n} = 0 \forall n = 1, \dots, K$ establishes that $J_p(\mathcal{I}(\omega, t))$ is singular everywhere on $\omega \times [0, T]$.

If. For an arbitrary $\mathbf{v} \in \mathbb{R}^K \setminus \{\mathbf{0}\}$ the non-singularity of Σ guarantees at least one non-zero entry for the vector $\Sigma \mathbf{v}$. Let it be the n th one, $\sigma_n^\top \mathbf{v} = \nu \neq 0$. Consider also the hyperplane $H_{\sigma_n} = \{\mathbf{x} \in \mathbb{R}^K : \sigma_n^\top \mathbf{x} = 0\}$ and the line $L(\mathbf{x}_0; \mathbf{v})$. As $\mathbf{v} \not\parallel H_{\sigma_n}$, \mathbb{R}^K can be spanned as $\cup_{\mathbf{x}_0 \in H_{\sigma_n}} L(\mathbf{x}_0; \mathbf{v})$. Hence, $\mathbf{v}^\top \mathbf{j}_{p,n} = \int_{H_{\sigma_n} \times H_{\sigma_n}} I(\mathbf{x}_0, \mathbf{y}_0; \mathbf{v}) d\mathbf{x}_0 d\mathbf{y}_0$. For any $\mathbf{x} \in L(\mathbf{x}_0; \mathbf{v})$ we now have $\mathbf{v}^\top \mathbf{x} = \mathbf{v}^\top \mathbf{x}_0 + t\mathbf{v}^\top \mathbf{v}$ and $\sigma_n^\top \mathbf{x} = t\nu \forall n = 1, \dots, K$. That is,

$$\begin{aligned}
& I(\mathbf{x}_0, \mathbf{y}_0; \mathbf{v}) \\
&= \mathbf{v}^\top (\mathbf{y}_0 - \mathbf{x}_0) \\
&\quad \int_{\mathbb{R}^2} u'(F(\mathbf{x}_0 + \tau\mathbf{v})) u'(F(\mathbf{y}_0 + t\mathbf{v})) e^{t\nu} e^{-\frac{(\mathbf{y}_0 + t\mathbf{v})^\top (\mathbf{y}_0 + t\mathbf{v}) + (\mathbf{x}_0 + \tau\mathbf{v})^\top (\mathbf{x}_0 + \tau\mathbf{v})}{2}} dt d\tau \\
&\quad + \mathbf{v}^\top \mathbf{v} \\
&\quad \int_{\mathbb{R}^2} u'(F(\mathbf{x}_0 + \tau\mathbf{v})) u'(F(\mathbf{y}_0 + t\mathbf{v})) e^{t\nu} (t - \tau) e^{-\frac{(\mathbf{y}_0 + t\mathbf{v})^\top (\mathbf{y}_0 + t\mathbf{v}) + (\mathbf{x}_0 + \tau\mathbf{v})^\top (\mathbf{x}_0 + \tau\mathbf{v})}{2}} dt d\tau \\
&= \mathbf{v}^\top (\mathbf{y}_0 - \mathbf{x}_0) \mathbb{E}_{(\mathbf{z}, \tilde{\mathbf{z}})} \left[e^{\frac{\nu(\tilde{z}_n - y_{0n})}{\nu}} u'(F(\mathbf{z})) u'(F(\tilde{\mathbf{z}})) \right] \\
&\quad + \mathbf{v}^\top \mathbf{v} \\
&\quad \int_{\mathbb{R}^2} u'(F(\mathbf{x}_0 + \tau\mathbf{v})) u'(F(\mathbf{y}_0 + t\mathbf{v})) e^{t\nu} (t - \tau) e^{-\frac{(\mathbf{y}_0 + t\mathbf{v})^\top (\mathbf{y}_0 + t\mathbf{v}) + (\mathbf{x}_0 + \tau\mathbf{v})^\top (\mathbf{x}_0 + \tau\mathbf{v})}{2}} dt d\tau
\end{aligned}$$

where $(\mathbf{z}, \tilde{\mathbf{z}}) \sim \mathcal{N}\left(-(\mathbf{x}_0, \mathbf{y}_0), \begin{pmatrix} \mathbf{v}\mathbf{v}^\top & \mathbf{0} \\ \mathbf{0} & \mathbf{v}\mathbf{v}^\top \end{pmatrix}\right)$.

There are two cases to consider. If \mathbf{v} is parallel to σ_n , then $\mathbf{v}^\top \mathbf{x}_0 = \mathbf{v}^\top \mathbf{y}_0 = 0$. Otherwise, we can write $H_{\sigma_n} = \cup_{r \in \mathbb{R}} H_{\sigma_n}^r$ where $H_{\sigma_n}^r = \{\mathbf{x}_0 \in H_{\sigma_n} : \mathbf{v}^\top \mathbf{x}_0 = r\}$. Which allows us to span \mathbb{R}^K as $\cup_{r \in \mathbb{R}} \cup_{\mathbf{x}_0 \in H_{\sigma_n}^r} L(\mathbf{x}_0; \mathbf{v})$ and write $\mathbf{v}^\top \mathbf{j}_{p,n} = \int_{\mathbb{R}} \left(\int_{H_{\sigma_n}^r \times H_{\sigma_n}^r} I(\mathbf{x}_0, \mathbf{y}_0; \mathbf{v}) d\mathbf{x}_0 d\mathbf{y}_0 \right) dr$. With respect to the integration in the brackets, in the expansion for $I(\mathbf{x}_0, \mathbf{y}_0; \mathbf{v})$ above we now have $\mathbf{v}^\top (\mathbf{y}_0 - \mathbf{x}_0) = r - r$. In either case, therefore, $\mathbf{v}^\top (\mathbf{y}_0 - \mathbf{x}_0) = 0$ and

$$\begin{aligned} & I(\mathbf{x}_0, \mathbf{y}_0; \mathbf{v}) \\ &= \mathbf{v}^\top \mathbf{v} \\ & \int_{\mathbb{R}^2} u'(F(\mathbf{x}_0 + \tau \mathbf{v})) u'(F(\mathbf{y}_0 + t \mathbf{v})) e^{t\nu} (t - \tau) e^{-\frac{(\mathbf{y}_0 + t\mathbf{v})^\top (\mathbf{y}_0 + t\mathbf{v}) + (\mathbf{x}_0 + \tau\mathbf{v})^\top (\mathbf{x}_0 + \tau\mathbf{v})}{2}} dt d\tau \\ &= \mathbf{v}^\top \mathbf{v} \\ & \int_{\mathbb{R}^2} u'(F(\mathbf{x}_0 + t \mathbf{v})) u'(F(\mathbf{y}_0 + \tau \mathbf{v})) e^{\tau\nu} (\tau - t) e^{-\frac{(\mathbf{y}_0 + \tau\mathbf{v})^\top (\mathbf{y}_0 + \tau\mathbf{v}) + (\mathbf{x}_0 + t\mathbf{v})^\top (\mathbf{x}_0 + t\mathbf{v})}{2}} dt d\tau \end{aligned}$$

Observe now that³⁶

$$I(\mathbf{x}_0, \mathbf{y}_0; \mathbf{v}) + I(\mathbf{y}_0, \mathbf{x}_0; \mathbf{v}) = \mathbf{v}^\top \mathbf{v} \int_{\mathbb{R}^2} g(\mathbf{x}_0 + \tau \mathbf{v}, \mathbf{y}_0 + t \mathbf{v}) (e^{t\nu} - e^{\tau\nu}) (t - \tau) dt d\tau$$

where $g : \mathbb{R}^{2K} \mapsto \mathbb{R}_{++}$ is given by

$$g(\mathbf{x}_0 + \tau \mathbf{v}, \mathbf{y}_0 + t \mathbf{v}) = u'(F(\mathbf{x}_0 + \tau \mathbf{v})) u'(F(\mathbf{y}_0 + t \mathbf{v})) e^{-\frac{(\mathbf{y}_0 + t\mathbf{v})^\top (\mathbf{y}_0 + t\mathbf{v}) + (\mathbf{x}_0 + \tau\mathbf{v})^\top (\mathbf{x}_0 + \tau\mathbf{v})}{2}}$$

Since $\mathbf{v}^\top \mathbf{v} > 0$ and $\nu (e^{t\nu} - e^{\tau\nu}) (t - \tau) > 0$, by Lemma A.4, $I(\mathbf{x}_0, \mathbf{y}_0; \mathbf{v}) + I(\mathbf{y}_0, \mathbf{x}_0; \mathbf{v})$ has the same sign as ν and so does $2\mathbf{v}^\top \mathbf{j}_{p,n}$. We have established, therefore, that, $\forall \mathbf{v} \in \mathbb{R}^K \setminus \{\mathbf{0}\}$, there exists at least one security n with $\mathbf{v}^\top \mathbf{j}_{p,n} \neq 0$. Clearly, $J_p(\mathcal{I}(\omega, t))$ is non-singular everywhere on $\Omega \times [0, T]$.

³⁶Here, I construct the sum $I(\mathbf{x}_0, \mathbf{y}_0; \mathbf{v}) + I(\mathbf{y}_0, \mathbf{x}_0; \mathbf{v})$ using the definition given by the first equality in the text for the first term of the sum and that given by the second equality for the second term.